Riemann Integration
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1 Riemann Integration

As we know, the first motivation of integration started with measuring some areas. Suppose that we want to measure the area enclosed by $y = \frac{1}{2}x^2$, $x = 2$, x-axis and y-axis. Our idea was *approximating* its area by some finite rectangles as the following figure.

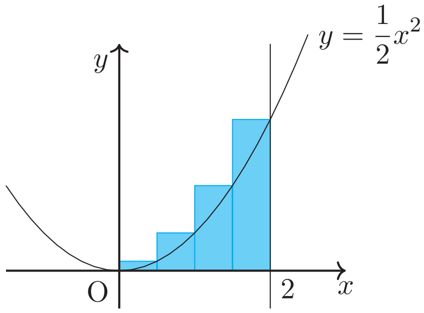

Figure 1

Given an interval $[a, b]$ and continuous real-valued function f, the classical approach of calculating

$$\int_a^b f(x)dx$$

is to divide the interval into n subintervals

$$[x_0, x_1], \cdots, [x_{n-1}, x_n]$$

where

$$x_1 - x_0 = \cdots = x_n - x_{n-1} = \Delta x \quad \text{and} \quad x_0 < x_1 < \cdots < x_n.$$

and to sample some value $x_i^* \in [x_i, x_{i+1}]$, so that we have the following estimate.

$$\sum_{i=0}^{n-1} \Delta x f(x_i^*) \approx \int_a^b f(x)dx$$

Taking the limit of $n \to \infty$, we have the elementary definition of the Riemann integration, that is,

$$\int_a^b f(x)dx = \sum_{i=0}^{\infty} \Delta x f^*(x_i).$$

In this section our goal is to explore how the Riemann integral is defined in the contemporary setting in $\mathbb{R}^d$. Starred(*) items deal with the case of $d \geq 2$, which may be omitted on first reading if you feel pressured.

1.1 Partitions, Upper Sums and Lower Sums

Definition 1 (*). **A rectangle**

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

is the Cartesian product of d closed and bounded intervals.

Definition 2. The length of the interval $[a, b] \subset \mathbb{R}$**,** denoted by $|[a, b]|$, is defined as $b - a$.

Definition 3 (*). **The volume of the rectangle** $R \subset \mathbb{R}^d$**,** denoted by $|R|$, is defined as

$$|R| = (b_1 - a_1) \times \cdots \times (b_d - a_d)$$

where $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$.

The first step to evaluate Riemann integrals is *partitioning* the given rectangular domain so that we may sample some value from each subrectangle.

Definition 4. For a given interval $[a, b]$, **a partition of** $[a, b]$ is a finite set

$$P = \{x_0, x_1, \cdots, x_n\}$$

where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

Each of the intervals $[x_i, x_{i+1}]$ is called **a subinterval determined by** P**.**

Figure 2: An evenly spaced partition of $[0, 3]$.

Figure 3: A partition of $[0, 3]$.

Note that the partition of an interval or a rectangle does not have to be evenly spaced. Moreover it seems natural to extend the definition of partition to $\mathbb{R}^d$ as follows.

Definition 5 (*). More generally, for a given rectangle $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$, **a partition of** R is an d-tuple

$$P = (P_1, \cdots, P_d)$$

where P_i is a partition of the interval $[a_i, b_i]$.

The rectangle

$$R' = I_1 \times \cdots \times I_d$$

is called **a subrectangle determined by** P where each I_i is a subinterval determined by P_i. We denote $\mathcal{P}(R, P)$ the set of all subrectangles of R determined by P.

Definition 6. For partitions

$$P = (P_1, \cdots, P_d) \quad \text{and} \quad P' = (P_1', \cdots, P_d')$$

of a rectangle R, P' is **a refinement of** P if $P_i' \subset P_i$ for each $1 \leq i \leq d$.

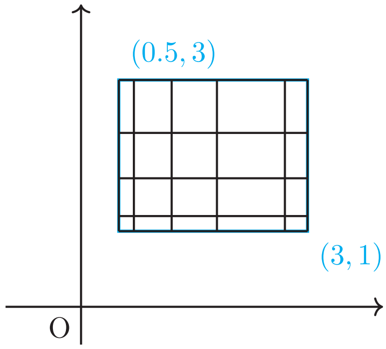

Figure 4: A partition of $[0.5, 3] \times [1, 3]$

Definition 7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P be a partition of $[a, b]$. The **upper sum** $U(f, P)$ and **the lower sum** $L(f, P)$ is defined as

$$U(f, P) = \sum_{i=0}^{n-1} \left(\sup_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i),$$
$$L(f, P) = \sum_{i=0}^{n-1} \left(\inf_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i).$$

Definition 8 (*)**.** Moreover, assume that $R \subset \mathbb{R}^d$ is a rectangle. Let $f : R \to \mathbb{R}$ be a bounded function and P be a partition of R. The **upper sum** $U(f, P)$ and **the lower sum** $L(f, P)$ is defined as

$$U(f, P) = \sum_{R' \in \mathcal{P}(R,P)} \left(\sup_{x \in R'} f(x) \right) |R'|,$$
$$L(f, P) = \sum_{R' \in \mathcal{P}(R,P)} \left(\inf_{x \in R'} f(x) \right) |R'|.$$

Proposition 9. Let $R \subset \mathbb{R}^d$ be a open and bounded set and P be a partition of R. If P' is a refinement of P, then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$$

for all bounded function $f : R \to \mathbb{R}$.

Proof. Obvious. □

1.2 Riemann Integrable

Definition 10. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. **An upper integral** of f and **a lower integral** of f over $[a, b]$ is defined as

$$\overline{\int_a^b} f = \sup_P U(f, P)$$

$$\underline{\int_a^b} f = \inf_P L(f, P)$$

where P is the partition of $[a,b]$. f is **Riemann-integrable** if

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

and we denote the common value of upper and lower integral by $\displaystyle\int_a^b f$ and $\mathcal{R}([a,b])$ the set of Riemann-integrable function on the interval $[a,b]$.

Indeed the same argument works in $\mathbb{R}^d$ with $d \geq 2$, from our observation that so an interval is a rectangle itself.

Definition 11 (*)**.** Let $R \subset \mathbb{R}^d$ a rectangle and $f : R \to \mathbb{R}$ be a bounded function. **An upper integral** of f and **a lower integral** of f over R is defined as

$$\overline{\int_a^b} f = \sup_P U(f,P)$$

$$\underline{\int_a^b} f = \inf_P L(f,P)$$

where P is the partition of $[a,b]$. f is **Riemann-integrable** on R if

$$\overline{\int_R} f = \underline{\int_R} f$$

and we denote the common value of upper and lower integral by $\displaystyle\int_R f$ and $\mathcal{R}(R)$ the set of Riemann-integrable function on the rectangle R.

Exercise 1. Define

$$f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1}{q} & \left(x \in \mathbb{Q} \setminus \{0,1\}, x = \dfrac{p}{q}, \gcd(p,q) = 1 \right) \\ 0 & \text{(otherwise)} \end{cases}$$

on $[0,1]$. Show that $f \notin \mathcal{R}([0,1])$ but $g \in \mathcal{R}([0,1])$.

Theorem 12. Suppose that $f, g \in \mathcal{R}([a,b])$ and $k \in \mathbb{R}$ then the followings are satisfied.

1. $kf \in \mathcal{R}([a,b])$, $\displaystyle\int_a^b kf = k\int_a^b f$.
2. $f+g \in \mathcal{R}([a,b])$, $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
3. If $f(x) \leq g(x)$ on $[a,b]$, then $\displaystyle\int_a^b f \leq \int_a^b g$.
4. If $f \in \mathcal{R}([b,c])$, then $f \in \mathcal{R}([a,c])$ and $\displaystyle\int_a^b f + \int_b^c f = \int_a^c f$.
5. $|f| \in \mathcal{R}([a,b])$, $\displaystyle\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof. The proof is rather elementary so we skip the details. □