A Brief Introduction to Real Analysis

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Preface

우울해서, 너무 숨이 막혀서, 도피하고 싶었기에 만들었습니다.
내일의 나는 오늘의 나보다는 나았으면.
가슴이 답답해서 숨을 쉴 수가 없다. 부디 이 집합들이 나를 치유하게 해 주소서.

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Chapter 1 | The Real Number System

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2.1 Open and Closed

2.1.1 Metric Space

Definition 2.1 (distance)**.** Distance is a real-valued function $d : X \times X \to \mathbb{R}_0^+$ such that

- $d \geq 0$, $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(y, z)$

Definition 2.2 (metric space)**.** For a set X, if the distance function d is defined on X, we call (X, d) a **metric space**. We also say that X is **metric**.

Definition 2.3 (an open ball centred at x and its radius r)**.**

$$B_{x,r} = \{y \in M \mid d(x, y) < r\}$$

2.1.2 Open and Closed Sets

Let (X, d) be a metric space.

Definition 2.4 (open set)**.** We say that $A \subset X$ is **open** if for every $x \in A$, there exists $r > 0$ such that $B_{x,r} \subset A$.

Definition 2.5 (closed set)**.** We say that $A \subset X$ is **closed** if A^C is open, i.e., for every $x \in A^C$, there exists $r > 0$ such that $B_{x,r} \subset A^C$.

Definition 2.6 (interior point)**.** We say that $x \in A$ is an **interior point** of A if there exists an open set V such that $x \in V \subset A$. We denote A^O a set of all interior points of A.

Union and Intersections

Theorem 2.7. Consider the collection of open sets $\{U_\alpha | \alpha \in I\}$, where I is an uncountable index set.

- $\bigcup_{\alpha \in I} U_\alpha$ is open.
- $\bigcap_{i=1}^{n} U_i$ is open.

Proof.

- For every $x \in \bigcup_{\alpha \in I} U_\alpha$, there exists $\beta \in I$ such that $x \in U_\beta$. Since U_β is open, there is $r > 0$ such that

$$B_{x,r} \subset U_\beta \subset \bigcup_{\alpha \in I} U_\alpha$$

, implying that $\bigcup_{\alpha \in I} U_\alpha$ is open.

- For every $x \in \bigcap_{i=1}^{n} U_i$, $x \in U_i$ for all $1 \leq i \leq n$. Since U_i is open for each i, there exists $r_i > 0$ such that

$$B_{x,r_i} \subset U_i.$$

Take $r = \min\{r_i : 1 \leq i \leq n\}$, then it follows that $B_{x,r} \subset U_i$ for every i, implying

$$x \in B_{x,r} \subset \bigcap_{i=1}^{n} U_i$$

, therefore $\bigcap_{i=1}^{n} U_i$ is open.

□

Theorem 2.8. Consider the collection of closed sets $\{V_\alpha | \alpha \in I\}$, where I is an uncountable index set.

- $\bigcap_{\alpha \in I} V_\alpha$ is closed.
- $\bigcup_{i=1}^{n} V_i$ is closed.

Proof. Trivial by De Morgan's Law. □

2.1.3 Limit Points

Let (X, d) be a metric space.

Definition 2.9 (limit point)**.** We say that x is an **limit point** of A whenever any open neighbourhood of x has a point in A other than x.
i.e., for every open neighbourhood V of x,

$$V \cap (M \backslash \{x\}) \neq \emptyset.$$

Definition 2.10. We denote A' a set of all limit points of A.

Definition 2.11 (isolated point)**.** We say that x is an **isolated point** of A if it is not a limit point of A.

Definition 2.12 (closed set)**.** We say that $A \subset X$ is **closed** if A contains all of the limit points of A, i.e.,

$$A' \subset A$$

.

Note that the definition above is **equivalent** to 2.5.

Proof. $(1 \implies 2)$ Suppose that $x \notin A$ is an limit point of A, then $x \in A^C$. Note that $M \backslash A$ is open since A is closed by our hypothesis. Therefore $\exists r > 0$ s.t. $B_{x,r} \subset A^C$ which is contrary to the fact that x is an limit point.
$(2 \impliedby 1)$ It is sufficient to show that A^C is open. Suppose that A^C is not open. Then $\exists x \in A^C$ s.t. any open neighbourhood containing x has an nonempty intersection with A, implying that $x \notin A$ but x is an limit point of A. This contradicts our hypothesis. □

Theorem 2.13. X and $\emptyset$ are both open and closed.

Proof. It is obvious that X is open $\implies$ $M^C = \emptyset$ is closed.
$\emptyset^C = X$ is closed since it contains all of the limit points of itself. $\implies$ $\emptyset$ is open. □

2.1.4 Open and Closed Relative

Let (X, d) be a metric space and $A \subset X$.

Definition 2.14 (open relative)**.** We say $U \subset A$ is **open relative** to A if there exists open set $V \subset X$ such that $U = V \cap A$.

Definition 2.15 (closed relative)**.** We say $U \subset A$ is **closed relative** to A if there exists closed set $F \subset X$ such that $U = F \cap A$.

2.1.5 Closure and Boundary

Definition 2.16 (closure)**.** We define the **closure** of A denoted by $\overline{A}$, in three ways:

- the intersection of all closed sets containing $A : (A_1)$
- the smallest closed set containing $A : (A_2)$
- $A \cup A' : (A_3)$

and these three definitions are **equivalent**.

Proof. $A_1 = A_2$: Obvious.
$(A_1 \supset A_3)$ Note that for collection $\{C_\alpha\}$ of all closed set containing A, each element satisfies that $C_\alpha \supset A$, $C_\alpha \supset B$. Hence $A_1 = \bigcap\limits_\alpha C_\alpha \supset A_3$
$(A_2 \subset A_3)$ Claim: $\forall x \in A_2$, if $x \notin A$, then $x \in A'$.
Suppose that $\exists x \in A_2$ s.t. $x \notin A$ and $x \notin B$, that is, x is not an limit point of A. Then, $\exists$ open neighbourhood V s.t. $x \in V$ and $V \cap (A\backslash \{x\}) = \emptyset \implies V \cap A = \emptyset$ since $x \notin A$. Then it follows that $A \subset V^C$ and V^C is closed. By the definition of A_2, we derive $x \in A_2 \subset V^c$, which contradicts our assumption. $\square$

Definition 2.17 (boundary)**.** We define the **boundary** of A as $\partial A = \text{bd}(A) = \overline{A} \cap \overline{A^C}$

Theorem 2.18. *$x \in \partial A$ iff. $\forall r > 0$, $B_{x,r} \cap A \neq \emptyset$ and $B_{x,r} \cap A^C \neq \emptyset$*

Proof. $(\impliedby)$ $\forall r > 0$, $B_{x,r} \cap A \neq \emptyset \implies x \in \overline{A}$ and $B_{x,r} \cap A^C \neq \emptyset \implies x \in \overline{A^C}$, therefore $x \in \overline{A^C} = \partial A$.
$(\implies)$ Consider the case of $x \in A$, then since $x \in \overline{A^C}$, x must be an limit point of A^C, implying $B_{x,r} \cap A^C \neq \emptyset$. Notice that it is trivial that $B_{x,r} \cap A \neq \emptyset$. WLOG, we can prove the case of $x \in A^C$. $\square$

2.1.6 Examples

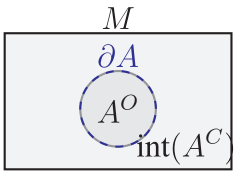
Here are some useful properties related to what we have learnt:
Let (X, d) be a metric space.

Example 2.19. 1. $(A^O)^C = \text{cl}(A^C)$

2. $(\overline{A})^C = \text{int}(A^C)$

3. $\partial A = \overline{A} \setminus \text{int}\, A$

We observe that A^O, ∂A, and $\text{int}(A^C)$ are a **partition** of X. i.e., these three sets are disjoint and $A^O \cup \partial A \cup \text{int}(A^C) = X$.



Proof. 1. $((A^O)^C \subset \overline{A^C})$ $\forall x \in (\text{int}\, A)^C$, $\forall r > 0$, $B_{x,r} \not\subset A \implies B_{x,r} \cap A^C \neq \emptyset$ which implies
$x \in A^C$ or $x \in (A^C)'$ therefore $x \in \text{cl}(A^C)$.

$((A^O)^C \supset \text{cl}(A^C))$ $\forall x \in \text{cl}(A^C)$, $x \in A^C$ or $x \in (A^C)'$. It suffices to consider the case of $x \in (A^C)'$. Then, $\forall r > 0$, $B_{x,r} \cap A^C \neq \emptyset \implies \nexists r > 0$ s.t. $B_{x,r} \subset A$, hence $x \in (A^O)^C$.

2.

$\square$

Example 2.20.

1. $\partial A \subset A^C \iff A$: open
2. $\partial A \subset A \iff A$: closed
3. $\partial A = \emptyset \iff$ A is both open and closed.

Example 2.21. A is bounded if and only if $\exists M \geq 0$ s.t. $\forall x, y \in A$, $d(x, y) \leq M$.

Proof. $\square$

Definition 2.22 (diameter)**.** We define **diameter of** A as

$$\operatorname{diam} A := \begin{cases} \sup\{d(x,y)\,|\,x,y \in A\} & (A \neq \emptyset) \\ 0 & (A = \emptyset) \end{cases}$$

2.2 Sequences and Series

2.2.1 Sequences

Let (X, d) be a metric space.

Definition 2.23 (convergence)**.** For the sequence $x_n \in X$, we say x_n **converges** to $x \in X$, or $\lim_{n\to\infty} x_n = x$ if for every open set V containing x, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies x_n \in V.$$

Note that $x_n \in X$ converges to $x \in X$ if and only if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies d(x_n, x) < \varepsilon.$$

Theorem 2.24. *$A \subset X$ is closed if and only if for every sequence $x_n \in A$ that converges in X, $\lim_{n\to\infty} x_n \in A$.*

Proof. ($\implies$) Suppose $\lim_{x\to\infty} x_n = x \notin A$, then $x \in A' \backslash A = \emptyset$ (contradiction)
($\impliedby$) It suffices to show that $A' \subset A$. Suppose that $x \in A'$, that is, $\forall$ open neighborhood V of x, $V \cap A \neq \emptyset$. For $\forall$ open neighborhood V of x, we can find $n \in \mathbb{N}$ s.t. $x_n \neq x$ and $x_n \in A$ implying that $x \in A$ hence A is closed. □

Theorem 2.25. *For a set $A \subset X$, $x \in \overline{A}$ iff. $\exists\, x_n \in A$ converges to x.*

Proof. It suffices to consider the case of $x \in \overline{A} \backslash A$. ($\implies$) $\forall x \in A'$, $\forall$ open neighborhood V of x, $V \cap A \neq \emptyset$. Thus we can construct $x_n \in A$ s.t. for $n \in \mathbb{N}$, $x_n \in B_{x,\frac{1}{n}} \cap A$ implying $x_n \to x$.
($\impliedby$) $\forall$ open neighborhood V of x, $n_0 \in \mathbb{N}$ s.t. $n_0 \geq n$ implies $x_n \in V$. Then $V \cap \{x_n \,|\, n \geq n_0\} \neq \emptyset \implies V \cap A \neq \emptyset$ which results that x is an limit point of A. □

Definition 2.26 (cluster point)**.** We say x a **cluster point** of x_n if $\forall\varepsilon > 0$, there exists infinitely many n such that $d(x_n, x) < \varepsilon$.

Theorem 2.27. x is a cluster point of x_n iff. there is a subsequence of x_n which converges to x.

Cauchy Sequences and Complete Metric Space

Let (X, d) be a metric space.

Definition 2.28 (Cauchy sequence)**.** For a metric space X, we say $x_n \in X$ a **Cauchy sequence** if $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $m, n \geq n_0$ implies $d(x_m, x_n) < \varepsilon$.

Definition 2.29. We say that a metric space X is **complete** if every Cauchy Sequence in X converges to a point in X.

Example 2.30. Show that every Cauchy sequence in metric space is convergent. Give an example of metric spaces X, $A \subset X$ such that Cauchy sequence $x_n \in A$ converges in X but not in A.

2.2.2 Spaces

Let K be a vector space.

Normed Spaces

Definition 2.31 (normed spaces)**.** A **norm** is a function $\|\| : K \to \mathbb{R}_0^+$ such that satisfies the followings:

1. $\|x\| \geq 0$ for $\forall x \in K$
2. $\|x\| = 0$ iff. $x = 0$
3. $\|\lambda x\| = |x| \, \|x\|$ for $\forall x \in K$ and scalar λ
4. $\|x + y\| \leq \|x\| + \|y\|$

We call $(K, \|\|)$ a **normed space**.

Theorem 2.32. A normed space is a metric space.

The proof is rather elementary so we left it as an exercise.

Inner Product Spaces

Definition 2.33 (inner product space)**.** An **inner product** is a function $\langle \cdot, \cdot \rangle : K \times K \to \mathbb{R}$ such that satisfies the followings:

1. $\langle x, x \rangle \geq 0$ for $\forall x \in K$
2. $\langle x, x \rangle = 0$ iff. $x = 0$
3. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for $\forall x, y \in K$ and $\lambda \in \mathbb{R}$.
4. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for $\forall x, y, z \in K$
5. $\langle x, y \rangle = \langle y, x \rangle$ for $\forall x, y \in K$

We call $(K, \langle \cdot, \cdot \rangle)$ an **inner product space**.

Cauchy-Schwarz Inequality 2.34. For an inner product space $(K, \langle \cdot, \cdot \rangle)$, $|\langle x, y\rangle| \leq (\langle x, x\rangle)^{\frac{1}{2}} (\langle y, y\rangle)^{\frac{1}{2}}$.

Proof. Assume that $x, y \neq 0$. For $\forall \alpha \in \mathbb{R}$, we have

$$0 \leq \langle \alpha x + y, \alpha x + y\rangle = \alpha^2 \langle x, x\rangle + 2\alpha \langle x, y\rangle + \langle y, y\rangle$$

Therefore we derive that $(\langle x, y\rangle)^2 - \langle x, x\rangle \langle y, y\rangle \leq 0$. □

Theorem 2.35. An inner product space is a normed space.

We can derive the triangle inequality from Cauchy-Schwarz inequality so that a norm is derived by inner product space.

2.2.3 Properties of Bounded and Cauchy Sequences

Definition 2.36 (bounded sequence)**.** For a normed space, we say a sequence x_n is **bounded** if $\exists X$ s.t. $\|x_n\| \leq X$ for $\forall n$.

Theorem 2.37. Every convergent sequence in a metric space is a Cauchy sequence.

Proof. Suppose x_n be a sequence s.t. $\lim_{n\to\infty} x_n = x$. For $\varepsilon > 0$, $\exists n_1 > 0$ s.t. $n \geq n_1$ implies $d(x, x_n) < \dfrac{\varepsilon}{2}$. Take $n_0 = n_1$, then $\forall X, n > n_0$, $d(x_X, x_n) \leq d(x_X, x) + d(x_n, x) < \epsilon$. □

Theorem 2.38. Every Cauchy sequence in a metric space is bounded.

Proof. Suppose x_n be a Cauchy sequence. Then, $\exists n_0$ s.t. $n, m \geq n_0$ implies $d(x_n, x_m) < 1$. Note that $d(x_n, x_{n_0}) < 1 \implies d(x_n, 0) < d(x_{n_0}) + 1$ for every $n \geq n_0$. Take $M = \max\{\|x_k\| \mid 1 \leq k \leq n_0\} + 1$, which follows that $\|x_n\| \leq M$ for all n. □

2.2.4 Series

2.3 Compactness

2.3.1 Compact Sets

Let (X, d) be a metric space.

Definition 2.39 (sequentially compact)**.** We say $A \subset X$ is **sequentially compact** if every sequence in A has a subsequence converges to a point in A.

Definition 2.40 (cover)**.** For a set $A \subset X$, we say a collection of sets $\{U_\alpha\}_{\alpha \in I}$ **a cover of** A if $A \subset \cup_{\alpha \in I} U_\alpha$.

Definition 2.41 (subcover)**.** We say a subcollection of a cover of A **a subcover of** A.

Definition 2.42 (open cover)**.** We call a cover **an open cover** if each element of it is open.

Definition 2.43 (compact)**.** $A \subset X$ is **compact** if every open cover of A has a finite subcover.

2.3.2 Lebesgue Number

Definition 2.44 (Lebesgue number)**.** We say $r > 0$ a **Lebesgue number** of A if for an open cover $\{U_\alpha\}_{\alpha \in I}$ of A, $\forall x \in A$, $B_{x,r} \subset U_k$ for some k .

Theorem 2.45 (existence of Lebesgue number)**.** If A is sequentially compact, then Lebesgue number r exists.

Proof. Suppose not, then $\exists$ seq $x_n \in A$ s.t. $B_{x_n, \frac{1}{n}} \notin U_k$ for any k. By our hypothesis that A is seq'ly cpt, $\exists$ subseq x_{n_k} of x_n s.t. $x_{n_k} \to x \in A$. Note that

- $x \in U_l$ for some $l \implies \exists r_1 > 0$ s.t. $x \in B_{x,r_1} \subset U_l$.
- since $x_{n_k} \to x$, $\exists k_0$ s.t. $k \geq k_0$ implies $x_{n_k} \in B_{x,\frac{r_1}{2}}$ and $\dfrac{1}{n_k} < \dfrac{r_1}{2}$.

Then we obtain

$$B_{x_{n_k}, \frac{1}{n_k}} \subset B_{x_{n_k}, \frac{r_1}{2}} \subset B_{x,r_1} \subset U_l$$

which contradicts our assumption. $\square$

2.3.3 Totally Bounded Sets

Definition 2.46 (totally bounded)**.** We say a set $A \subset X$ is **totally bounded** if $\forall r > 0$, there exists finite set $\{x_1, x_2, \cdots, x_n\} \subset K$ such that $A \subset \bigcup_{i=1}^{n} B_{x_i,r}$.

Theorem 2.47. If A is sequentially compact, then A is totally bounded.

Proof. Suppose not. Let $r > 0$ be given. We can construct seq $x_n \in A$ by

- taking some $x_1 \in A$ and
- $\forall k \in \mathbb{N}$, choosing x_{k+1} s.t. $d(x_{k+1}, x_m) > r$ for every $m \leq k$

Such x_{k+1} exists for every k because A is not totally bdd, which follows that it is unable to cover A by finite numbers of open balls with radius r. Suppose that subseq x_{n_k} of x_n converges, then $\forall r_1 > 0$, $\exists n_0$ s.t. $n \geq n_0$ implies $x_n \in B_{x,r_1}$. Take $r_1 = \frac{r}{2}$ then it contradicts our supposition by our construction of x_n. Thus we conclude that

- $\nexists$ convergent subseq x_{n_k} of x_n

, which contradicts our assumption. $\square$

2.3.4 Bolzano-Weierstrass Theorem

Theorem 2.48 (Bolzano-Weierstrass)**.** A subset of metric space is compact if and only if it is sequentially compact.

Proof. ($\implies$) Suppose not, then $\exists$ seq x_n s.t. $\nexists$ convergent subseq x_{n_k}. $\square$

Theorem 2.49. A metric space X is compact iff. it is complete and totally bounded.

Proof. ($\Longrightarrow$) Since X is cpt, it is seq'ly cpt, which follows that X is totally bdd. Note that every Cauchy seq x_n in X is convergent, which implies $x_n \to x \in X$ by our hypothesis that X is seq'ly cpt. Therefore X is complete.
($\Longleftarrow$) It suffices to show that X is seq'ly cpt. For every sequence $x_n \in X$, WLOG, it is enough to assume that $x_i \neq x_j$ if $i \neq j$. We want to find some convergent subseq of x_n. Since X is totally bounded, $\forall m \in \mathbb{N}$, we can construct some finite set $A_m = \{y_1^m, y_2^m, \cdots\}$ such that $X \subset \bigcup_{y \in A_m} B_{y, \frac{1}{m}}$. For each m, we can choose a_m s.t. $\exists \infty x_n \in B_{y_{m,a_m}, \frac{1}{m}}$ and $\bigcap_{i=1}^{m} B_{y_{i,a_i}, \frac{1}{i}} \neq \emptyset$. Take $x_{n_m} \in B_{y_{m,k}}$ (미완성) $\square$

2.3.5 Heine-Borel Theorem

Theorem 2.50 (Heine-Borel theorem)**.** $A \subset \mathbb{R}^n$ is compact iff. A is closed and bounded.

2.3.6 Nested Set Property

Definition 2.51 (finite intersection property)**.** In a metric space X, we say a collection $\{V_\alpha\}$ of closed sets in X has the **finite intersection property** if the intersection of the any finite number of V_α with A is nonempty. i.e.,

$$\bigcap_{i=1}^{n} V_i \neq \emptyset$$

Theorem 2.52. A is compact iff. for all collection $\{V_\alpha\}$ of finite intersection property, $A \cap \bigcap_{\alpha \in I} V_\alpha \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose not, that is, $A \cap V_\alpha = \emptyset$. Let $U_\alpha := V_\alpha^C$, then $A \subset \bigcup_{\alpha \in I} U_\alpha$. Since A is compact, $\exists$ finite subcover $\{U_{\alpha_i}\}_{i=1}^n$ of A, implying $A \cap \bigcap_{i=1}^{n} V_{\alpha_i} = \emptyset$ which violates the finite intersection property.
($\Leftarrow$) For an open cover $\{U_\alpha\}_{\alpha \in I}$ of A, let $V_\alpha := U_\alpha$ □

2.4 Connectedness

2.4.1 Path-Connected Sets

Definition 2.53 (path-connected)**.** We say that a set A is **path-connected** if $\forall x, y \in A$, $\exists$ continuous function $f : [0, 1] \to A$ s.t. $f(0) = x$ and $f(1) = y$.

2.4.2 Connected Sets

Definition 2.54 (disconnected)**.**

- We say a set A is **disconnected** if $\exists$ two open sets U, V satisfying these properties:
  - $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$
  - $A \subset U \cup V$
  - $(U \cap A) \cap (V \cap A) = \emptyset$

or, equivalently,

- We say a set A is **disconnected** if $\exists$ two disjoint open sets U, V satisfying these properties:
  - $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$
  - $A \subset U \cup V$

in these case, we say U and V separate A.

Definition 2.55 (connected)**.** We say a set A is **connected** if it is not disconnected.

Theorem 2.56. Path-connectedness implies connectedness, i.e., if a set A is path-connected then A is connected.

Proof.

Lemma 2.57. $[0, 1]$ is connected.

Proof. Suppose not, then $\exists$ open sets U, V such that

1. $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$
2. $A \subset U \cup V$
3. $(U \cap A) \cap (V \cap A) = \emptyset$

WLOG, it suffices to consider the case of $0 \in U$ and $1 \in V$. Denote

$$c := \sup \{x \in [0, 1] \,|\, x \in U \cap A\}$$

, then it follows that $c \notin U$ and $c \notin V$ which leads to contradiction. □

Suppose not, then $\exists$ open sets U, V disconnects A. Choose $x \in U \cap A$, $y \in V \cap A$ and we could construct function $f : [0, 1] \to A$ such that $f(0) = x$ and $f(1) = y$. □

Chapter 3 | Continuous Mappings

3.1 Continuity

3.2 Uniform Continuity

Chapter 4 | Differentiable Mappings

Chapter 5 | Riemann-Stieltjes Integrals

5.1 Riemann Integration of Functions of One Variable

5.1.1 Partitions, Upper Sums and Lower Sums

Definition 5.1 (partitions)**.** For a given interval $[a, b]$, we say the finite set

$$\mathcal{P} = \{x_0, x_1, \cdots, x_n\}, \quad a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

a **partition** of $[a, b]$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P}$ be a partition of $[a, b]$.

Definition 5.2 (upper sums)**.** We define a **upper sum** $\mathcal{U}_{\mathcal{P}} = \mathcal{U}(f, \mathcal{P})$ as

$$\mathcal{U}_{\mathcal{P}} = \sum_{i=0}^{n-1} \left(\sup_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)$$

Definition 5.3 (lower sums)**.** We define a **lower sum** $\mathcal{L}_{\mathcal{P}} = \mathcal{L}(f, \mathcal{P})$ as

$$\mathcal{L}_{\mathcal{P}} = \sum_{i=0}^{n-1} \left(\sup_{x \in [x_i, x_{i+1}]} f(x) \right) (x_{i+1} - x_i)$$

Definition 5.4. For partitions $\mathcal{P}$ and $\mathcal{P}'$ of $[a, b]$, we say $\mathcal{P}'$ is **finer** than $\mathcal{P}$ if

$$\mathcal{P} \subset \mathcal{P}'$$

or, we say $\mathcal{P}'$ a **refinement** of $\mathcal{P}$.

Theorem 5.5. If $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then

$$\mathcal{L}_{\mathcal{P}} \le \mathcal{L}'_{\mathcal{P}} \le \mathcal{U}'_{\mathcal{P}} \le \mathcal{U}_{\mathcal{P}}$$

Proof. Trivial. □

5.1.2 Riemann Integrable

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Definition 5.6. We say

$$\overline{\int_a^b} f = \sup \{\mathcal{U}_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

an **upper integral** of f and

$$\underline{\int_a^b} f = \inf \{\mathcal{U}_{\mathcal{P}} \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$

a **lower integral** of f over $[a, b]$.

Definition 5.7. f is **Riemann-integrable** if

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

and we denote the common value of upper and lower integral by $\int_a^b f$ and $\mathscr{R}$ the set of Riemann-integrable function.

Theorem 5.8. Suppose that $f, g \in \mathscr{R}$ on $[a, b]$, then

1. $\forall k \in \mathbb{R}$, $kf \in \mathscr{R}$ and $\int_a^b kf = k \int_a^b f$.
2. $f + g \in \mathscr{R}$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
3. $\forall x \in [a, b]$, if $f(x) \leq g(x)$, then $\int_a^b f \leq \int_a^b g$.
4. Assume that $f \in \mathscr{R}$ on $[b, c]$, then $f \in \mathscr{R}$ in $[a, c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.
5. Assume that $|f| \in \mathscr{R}$, then $\left| \int_a^b f \right| \leq \int_a^b |f|$.

Proof. The proof is rather elementary so we skip the details. $\square$

Theorem 5.9. if $f : [a, b] \to \mathbb{R}$ is bounded and continuous except for finitely many points in $[a, b]$, then $f \in \mathscr{R}$ on $[a, b]$

Theorem 5.10. if $f : [a, b] \to \mathbb{R}$ is bounded and monotonous, then $f \in \mathscr{R}$ on $[a, b]$

5.2 Fundamental Theorem of Calculus

Theorem 5.11 (fundamental theorem of calculus).

Chapter 6 | Sequences and Series of Functions

6.1 Uniform Convergence

6.1.1 Convergence of a Sequence of Functions

Let $f_n : A \to N$ be a sequence of functions, where (N, ρ) is a metric space.

Definition 6.1 (pointwise convergence)**.** We say f_n **converges pointwisely** to f if for each $x \in A$, $\forall \varepsilon > 0$, $\exists n(x) \in \mathbb{N}$ such that $n \geq n(x)$ implies $\rho(f_n(x), f(x)) < \varepsilon$. And we denote

$$f_n \to f \quad \text{pointwisely.}$$

Definition 6.2 (uniform convergence)**.** We say f_n **converges uniformly** to f if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\rho(f_n(x), f(x))$ for every $x \in A$. And we denote

$$f_n \to f \quad \text{uniformly.}$$

Theorem 6.3. Suppose that $f_n \to f$ uniformly, then f is continuous.

Proof.

Claim. $\forall x, y \in A$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$.

$$\rho(f(x), f(y)) \leq \rho\left(f(x), f_n(x)\right) + \rho\left(f_n(x), f_n(y)\right) + \rho\left(f(y), f_n(y)\right) < \epsilon$$

$\square$

6.1.2 Cauchy Criterion

Theorem 6.4 (Cauchy criterion)**.** $f_n : A \to N$ converges uniformly if and only if $\forall \varepsilon > 0$, $\exists n_0$ such that

$$m, n \geq n_0 \implies \rho(f_n(x), f_m(x)) < \varepsilon \; \forall x \in A.$$

Proof. □

6.1.3 Weierstrass M Test

Theorem 6.5 (Weierstrass M test)**.** Further assume that N is a complete normed space and $g_n : A \to N$.

If $\|g_n(x)\| < M_n \; \forall x \in A$ and $\sum_{n=0}^{\infty} M_n < \infty$, then $\sum_{n=0}^{\infty} g_n$ converges uniformly.

6.1.4 Properties of Uniform Convergence

Theorem 6.6. Suppose that $f_n \in \mathscr{R}([a,b])$. If $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathscr{R}([a,b])$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. □

Theorem 6.7. Suppose that $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly, then integration and summation can be interchanged, i.e.,

$$\int_a^b \sum_{n=1}^{\infty} g_n(x)dx = \sum_{n=1}^{\infty}$$

Theorem 6.8. Suppose that $f_n \in \mathcal{C}^1((a,b),\mathbb{R})$. If

$$f_n \to f \text{ pointwisely} \quad \text{and} \quad f_n' \to g \text{ uniformly}$$

, then f is differentiable and $f = g$.

Proof. □

Theorem 6.9. Suppose that $g_n \in \mathcal{C}^1$. If $\sum_{n=1}^{\infty} g_n$ converges pointwisely and $\sum_{n=1}^{\infty} g_n'$ converges uniformly, then

$$\left(\sum_{n=1}^{\infty} g_n(x)\right)' = \sum_{n=1}^{\infty} g_n'(x).$$

Proof. □

6.2 Spaces of Continuous Functions

Let (M, d) be a metric space and $(N, \rho, \|\|)$ be a complete normed space.
We denote $\mathcal{C} = \{f : A \subset M \to N \mid f \text{ is continuous}\}$ and $\mathcal{C}_b = \left\{ f \in \mathcal{C} \,\middle|\, \sup_{x \in A} |f(x)| < \infty \right\}$.

Definition 6.10. We say the collection of functions $\Pi = \{f_\alpha \mid \alpha \in I\}$ is **equicontinuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\rho(f(x), f(y)) < \varepsilon$ if $d(x, y) < \delta$ for every $f \in \Pi$.

6.2.1 Arzela-Ascoli Theorem

Theorem 6.11 (Arzela-Ascoli)**.** Let $A \subset M$ be compact, then $\mathcal{B} \subset \mathcal{C}_b$ is compact if and only if it is closed, pointwise compact, and equicontinuous.

Proof. ($\Longleftarrow$) By the Volzano-Weierstrass theorem, it suffices to show that $\mathcal{B}$ is sequentially compact.

Claim. For every sequence of functions $f_n \in \mathcal{B}$, there exists convergent subsequence of f_n.

Since A is compact, it is totally bounded, thus $\forall \delta > 0$, there is a finite set

$$C_\delta = \{y_{\delta_1}, y_{\delta_2}, \cdots, y_{\delta_n}\} \tag{6.1}$$

such that $\bigcup_{i=1}^{n} B_{y_{\delta_i}, \delta} \supset A$. Let

$$C = \bigcup_{i=1}^{n} C_{\frac{1}{n}}$$

and since C is countable, we relabel it as $C = \{x_1, x_2, \cdots\}$. For a sequence of functions $f_n \in \mathcal{B}$, since $\mathcal{B}$ is pointwisely compact, we can construct a subsequence $f_{1,j}$ of f such that $f_{1,j}(x_1)$ converges. Inductively, $\forall k \in \mathbb{N}$ we construct a subsequence $f_{k+1,j}$ of $f_{k,j}$ such that $f_{k+1,j}(x_{k+1})$ converges.
Define

$$g_n = f_{n,n}$$

, then we observe $g_n(x_i)$ converges $\forall i \in \mathbb{N}$.
It is enough to finish our proof by showing that g is uniformly convergent, i.e.,

Claim. For each $x \in A$, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ implies $\rho(g_n(x), g_m(x)) < \varepsilon$.

Since g_n is equicontinuous, we could choose $\delta > 0$ such that $\forall i$, $d(x, y) < \delta$ implies

$$\rho(g_i(x), g_i(y)) < \frac{\varepsilon}{3}.$$

Depending to our choice of δ, we construct a finite set C_δ as 6.1. For a given x, take $y \in C_\delta$ such that $d(x, y) < \delta$. Since g_i is convergent pointwisely, take $n_0 \in \mathbb{N}$ such that $n, m \geq n_0$ implies

$$\rho(g_n(y), g_m(y)) < \frac{\varepsilon}{3}.$$

Then, by the triangle inequality, we obtain

$$\rho(g_n(x), g_m(x)) \leq \rho(g_n(x), g_n(y)) + \rho(g_n(y), g_m(y)) + \rho(g_m(y), g_m(x)) < \varepsilon.$$

$(\Longrightarrow)$ $\square$

Theorem 6.12. Let $N = \mathbb{R}^d$. Assume that $\mathcal{B} \subset \mathcal{C}\left(A, \mathbb{R}^d\right)$ is equicontinuous and pointwise bounded. Then every sequence in $\mathcal{B}$ has a uniformly convergent subsequence.

6.3 Contraction Mappings

6.3.1 Contraction Mapping Principle

Let (M, d) be a complete metric space.

Theorem 6.13 (contraction mapping principle)**.** For the mapping $\Phi : M \to M$, if there exists a constant $k \in [0, 1)$ such that

$$\forall x, y \in M,\ d(\Phi(x), \Phi(y)) \leq kd(x, y)$$

, then there is a unique fixed point x_*. i.e.,

$$\exists! x_* \in M \quad \text{such that} \quad \Phi(x_*) = x_*.$$

Further assume that $x_0 \in M$ and $\forall n \in \mathbb{N}$, $\Phi(x_{n-1}) = x_n$, then

$$\lim_{n\to\infty} x_n = x_*.$$

Proof. (Uniqueness) Suppose there exists y_* is another fixed point of Φ, then

$$d(\Phi(x_*), \Phi(y_*)) = d(x_*, y_*) \leq kd(x_* y_*) \implies (1-k)d(x_*, y_*) \leq 0$$

therefore $x_* = y_*$ (contradiction).
(Existence)

Claim. x_n is Cauchy.
For a given $\varepsilon > 0$, take $n_0 = \min\left\{ n \,\middle|\, \frac{k^n}{1-k} d(x_0, x_1) < \varepsilon \right\}$, then

$$m > n \geq n_0 \implies d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-n-1} k^{n+i} d(x_0, x_1) \leq \frac{k^n}{1-k} d(x_0, x_1) < \varepsilon.$$

Therefore the limit $\lim_{n\to\infty} x_n = x_*$ exists. $\square$

Theorem 6.14. Let $f : \mathbb{R}^2 \rightarrow M$ be defined in a neighbourhood at $(t_0, x_0) \in \mathbb{R}^2$ and satisfying the following Lipschitz condition:
$\exists K > 0$ such that

$$|f(t,x) - f(t,y)| \leq K|x-y|$$

for all x, y in the neighbourhood of (t_0, x_0).
We consider

$$\frac{dx}{dt} = f(t,x), \quad x\left(t_0\right) = x_0 \tag{6.2}$$

Under the above assumptions, the above differential equation 6.2 has a unique $\mathcal{C}^1$ solution

$$x = \Phi(t)$$

with $x_0 = \Phi\left(t_0\right)$ for $t \in (t_0 - \delta, t + \delta)$, i.e.

$$\phi'(t) = f(t, \phi(t))$$

Proof. □

6.3.2 Fredholm Equations

Definition 6.15 (Fredholm equations)**.** We say the integral equation of the form

$$f(x) = \lambda \int_a^b K(x,y)f(y)dy + \varphi(x) \tag{6.3}$$

Fredholm equation.

Theorem 6.16. Assume that K and φ are continuous, then we have $|K(x,y)| < M$ on $[a,b] \times [a,b]$. If $\lambda M|b-a| < 1$, then the above Fredholm equation 6.3 has a unique solution.

Proof. □

6.3.3 Volterra Integral Equations

Definition 6.17 (Volterra equations)**.** We say the integral equation of the form

$$f(x) = \lambda \int_a^x K(x,y)f(y)dy + \varphi(x) \tag{6.4}$$

Volterra equation.

Theorem 6.18. Assume that K and φ are continuous, then the above Volterra equation 6.4 has a unique solution for any λ.

6.4 Series and Approximations

6.4.1 Bernstein Polynomials

Let $f \in \mathcal{C}([0,1],\mathbb{R})$.

Definition 6.19. We define the sequence of Bernstein polynomials

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Theorem 6.20. For a given $\varepsilon > 0$, there exists a polynomial $p(x)$ such that

$$\|p - f\| < \varepsilon.$$

Furthermore, the sequence of Bernstein polynomials

$$p_n(x) \to f \quad \text{uniformly}.$$

6.4.2 Stone-Weierstrass Theorem

Let (M, d) be a metric space and $\mathcal{A} \subset \{f : A \to \mathbb{R}\}$.

Definition 6.21 (algebra)**.** We say $\mathcal{A}$ is an algebra if $\forall f, g \in \mathcal{A}$ and $\forall \alpha \in \mathbb{R}$, $f + g, fg, \alpha f \in \mathcal{A}$, that is, $\mathcal{A}$ is closed under addition, multiplication, and scalar multiplication.

Definition 6.22 (separates points)**.** We say $\mathcal{A}$ separates points on A if $\forall x, y \in A$, if $x \neq y$, then $\exists f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 6.23 (Stone-Weierstrass)**.** Let $A \subset M$ be compact and $\mathcal{B} \subset \mathcal{C}(A, \mathbb{R})$ satisfy the following:

- $\mathcal{B}$ is an algebra.
- The nonzero constant function $1 \in \mathcal{B}$. ($\iff \forall x \in A$, $\exists f \in \mathcal{B}$ such that $f(x) \neq 0$)
- $\mathcal{B}$ separates points on A.

Then, $\mathcal{B}$ is dense in $\mathcal{C}(A, \mathbb{R})$, i.e., $\overline{\mathcal{B}} = \mathcal{C}(A, \mathbb{R})$.

Proof. We first introduce these lemmas before starting the proof.

Lemma 6.24. If $f \in \overline{\mathcal{B}}$, then $|f| \in \overline{\mathcal{B}}$.

Proof. Obvious. □

Lemma 6.25. If $f, g \in \overline{\mathcal{B}}$, then $\max\{f, g\}, \min\{f, g\} \in \mathcal{B}$.

Proof. This is direct from our previous result. □

For the preparation, $\forall x_1, x_2 \in A$, we define

$$f_{x_1,x_2}(x) = \frac{h(x_1) - h(x_2)}{g(x_1) - g(x_2)} g(x) + \frac{g(x_1)h(x_2) - h(x_1)g(x_2)}{g(x_1) - g(x_2)}$$

so that

$$f_{x_1,x_2}(x_1) = h(x_1), \quad f_{x_1,x_2}(x_2) = h(x_2).$$

From our construction, we observe that for a given x and $\forall y \in A \setminus \{x\}$, a function $f_{y,x}(z)$ satisfies

$$f_{y,x}(x) = h(x), \quad f_{y,x}(x) = h(y).$$

To finish the proof, it suffices to show that the following claim is true.

Claim. For a given function $h \in \mathcal{C}(A, \mathbb{R})$, $\exists f \in \overline{\mathcal{B}}$ such that $f = h$.

For a given $\varepsilon > 0$, there exists open neighbourhood U_y of y such that

$$z \in U_y \implies f_{y,x}(z) > h(z) - \varepsilon.$$

Note that $\{U_y \mid y \in A\}$ is an open cover of A and there exists its finite subcover

$$\{U_{y_1}, U_{y_2}, \cdots, U_{y_n}\}$$

which covers A.

Define

$$f_x(z) = \max\{f_{y_i,x} \mid 1 \leq i \leq n\}$$

, then

$$f_x \in \overline{\mathcal{B}} \text{ (by the lemma 6.25)}, \quad \forall z \in A,\ f_x(z) \geq h(z) - \varepsilon \quad \text{and} \quad f_x(x) = h(x).$$

On the other hand, for a given $\varepsilon > 0$, there exists open neighbourhood V_x of x such that

$$z \in V_x \implies f_x(z) < h(z) + \varepsilon.$$

Note that $\{V_x \mid x \in A\}$ is an open cover of A and there exists its finite subcover

$$\{V_{x_1}, V_{x_2}, \cdots, V_{x_m}\}$$

which covers A.

Define

$$f_x(z) = \min\{f_{x_i} \mid 1 \leq i \leq m\}$$

, then

$$f \in \overline{\mathcal{B}} \text{ (by the lemma 6.25)} \quad \text{and} \quad \forall x \in A,\ |f(z) - h(z)| < \varepsilon$$

which is enough to finish our proof. □

6.4.3 Abel's Test

Theorem 6.26 (Abel's partial summation formula)**.** Denote $s_n = \sum_{k=1}^{n} a_k$, then

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{k+1} - b_k) \\ &= s_n b_1 + \sum_{k=1}^{n} (s_n - s_k)(b_{k+1} - b_k)\end{aligned}$$

Proof. The proof is so elementary that we left it as an exercise. □

Theorem 6.27 (Abel's test)**.** Let $A \in \mathbb{R}^d$ and $\varphi_n : A \to \mathbb{R}$ be a uniformly bounded and decreasing sequence of functions. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then so does $\sum_{n=1}^{\infty} \varphi_n(x) f_n(x)$.

Proof. Denote $M = \sup_{x\in A} |\varphi_n(x)|$, $s_n(x) = \sum_{k=1}^{n} f_k(x)$ and $r_n(x) = \sum_{k=1}^{n} \varphi_k(x) f_k(x)$. By the Abel's partial summation formula, we obtain for $m < n$

$$r_n(x) - r_m(x) = (s_n(x) - s_m(x))\, \varphi_{m+1}(x) + \sum_{k=m+1}^{n} (s_n(x) - s_m(x))\, (\varphi_{k+1}(x) - \varphi_k(x))\,.$$

For a given $\varepsilon > 0$, since $s_n(x)$ converges uniformly, there exists $n_0 \in \mathbb{N}$ such that

$$n, m \geq n_0 \implies \forall x \in A,\ |s_n(x) - s_m(x)| < \frac{\varepsilon}{3M}.$$

Note that

$$\begin{aligned}&\sum_{k=m+1}^{n} (s_n(x) - s_m(x))\, (\varphi_{k+1}(x) - \varphi_k(x)) \\ &= \sum_{k=m+1}^{n} \frac{\varepsilon}{3M} (\varphi_k(x) - \varphi_{k+1}(x)) + \sum_{k=m+1}^{n} \left(\frac{\varepsilon}{3M} + s_n(x) - s_m(x)\right) (\varphi_{k+1}(x) - \varphi_k(x)) \\ &\leq \sum_{k=m+1}^{n} \frac{\varepsilon}{3M} (\varphi_k(x) - \varphi_{k+1}(x))\end{aligned} \tag{6.5}$$

□

6.4.4 Dirichlet's Test

Theorem 6.28. For sequences $f_n : A \subset \mathbb{R}^m \to \mathbb{R}$ and $g_n : A \subset \mathbb{R}^m \to \mathbb{R}$, if $\exists M > 0$,

$$\sup_{x \in A} \left| \sum_{k=1}^{n} f_k(x) \right| \leq M \quad \forall n \in \mathbb{N}$$

and $g_n(x)$ is nonnegative and nonincreasing sequence of functions, i.e.,

$$g_{n+1}(x) \leq g_n(x) \quad \text{and} \quad g_n \geq 0$$

such that

$$g_n \to 0 \quad \text{uniformly}$$

, then $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on A.

Chapter 7 | Special Functions and Summability of Series

7.1 Power Series

7.2 Summability of Series

7.2.1 Cesaro Summability

Definition 7.1. Set $S_n = \sum_{k=1}^{n} a_k$ and $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} S_k$. If $\lim_{n\to\infty} \sigma_n = A$, then we say that the series $\sum_{k=1}^{\infty} a_k$ is called Cesaro 1 -summable or $(\,\mathrm{C}, 1\,)$ summable to A, and denote $\sum_{k=1}^{\infty} a_k = A(\mathrm{C}, 1)$

7.2.2 Abel Summability

Definition 7.2. $\sum_{k=1}^{\infty} a_k$ is is summable in the sense of Abel if $\lim_{x \to 1^-} \sum_{k=1}^{\infty} a_k x^k = A$, and we denote $\sum_{k=1}^{\infty} a_k = A$ (Abel).

Note that

$$\text{summable} \implies (C,1) \text{ summable} \implies \text{Abel summable}$$

and the inverse is not true in general.

Chapter 8 | The Lebesgue Theory

Chapter 9 | Functions of Several Variables

Chapter 10 | Vector Analysis

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