sukita1729@gmail.com

§ 11.1 Sequences

◎ Defn> A sequence is a function defined on $\mathbb{N}$, whose range is contained in $\mathbb{R}$.
$f : \mathbb{N} \to \mathbb{R}$ or $f : \mathbb{N} \to \mathbb{C}$, $f(n) := a_n$.

◎ Defn> A sequence $\{a_n\}$ has the limit L $\Leftrightarrow$ $\lim_{n\to\infty} a_n = L$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.
$n > N \Rightarrow |a_n - L| < \varepsilon$

◎ Defn> $\lim_{n\to\infty} a_n = \infty \Leftrightarrow \forall M > 0,\ \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow a_n > M$

◎ Thm> If $\lim_{x\to\infty} f(x) = L$ and $f(x) = a_n$, then $\lim_{n\to\infty} a_n = L$.

pf) Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = L$, $\exists M > 0$ s.t. $x > M \Rightarrow$ $|f(x) - L| < \varepsilon$. Take $N = [M] + 1 \in \mathbb{N}$ then $M < [M] + 1 = N$. If $n > N$ then $n > N > M \Rightarrow |f(x) - L| < \varepsilon$, $|a_n - L| < \varepsilon$, $\lim_{n\to\infty} a_n = L$ ■

ex) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ $(\because \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0)$

◎ Thm> Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then the following property holds.

① $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

② $\lim_{n\to\infty} (c a_n) = c \lim_{n\to\infty} a_n$ $(c \in \mathbb{R})$

③ $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

④ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ $(b_n \neq 0,\ \lim_{n\to\infty} b_n \neq 0)$

pf) ④ Since $\lim_{n\to\infty} b_n = M \neq 0$ and $\frac{|M|}{2} > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow$ $|b_n - M| < \frac{|M|}{2} \Rightarrow \big||b_n| - |M|\big| \leq |b_n - M| < \frac{|M|}{2} \Rightarrow$ $\frac{|M|}{2} < |b_n| < |M| + \frac{|M|}{2} \Rightarrow \frac{1}{|b_n|} < \frac{2}{|M|}$. Then for $n > N_1$, we have

$$\left|\frac{a_n}{b_n} - \frac{L}{M}\right| = \frac{Ma_n - Lb_n}{Mb_n} = \frac{|Ma_n - Lb_n - ML + ML|}{|M||b_n|}$$

$$= \frac{1}{|M||b_n|}|M(a_n - L) + L(M - b_n)| \le \frac{2}{M^2}(|M||a_n - L| + |L||b_n - M|)$$

$$= \frac{2}{|M|}|a_n - L| + \frac{2|L|}{M^2}|b_n - M|.$$

Since $\lim_{n\to\infty} a_n = L$ and $\frac{|M|}{4}\varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow |a_n - L| < \frac{|M|}{4}\varepsilon$

Since $\lim_{n\to\infty} b_n = M$ and $\frac{M^2\varepsilon}{4(|L|+1)} > 0$, $\exists N_3 \in \mathbb{N}$ s.t. $n > N_3 \Rightarrow$

$|b_n - M| < \frac{M^2\varepsilon}{4(|L|+1)}$

Therefore, take $N := \max\{N_1, N_2, N_3\}$ then $n > N \Rightarrow$

$$\left|\frac{a_n}{b_n} - \frac{L}{M}\right| \le \frac{2}{|M|}|a_n - L| + \frac{2|L|}{M^2}|b_n - M| < \frac{2}{|M|}\times\frac{|M|}{4}\varepsilon + \frac{2|L|}{M^2}\times\frac{M^2\varepsilon}{4(|L|+1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacksquare$$

◎ Thm> ① $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ $(p, n > 0)$

② $\lim_{n\to\infty} |a_n| = 0 \Leftrightarrow \lim_{n\to\infty} a_n = 0$

③ $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} |a_n| = |L|$

ex) $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ since $\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$

◎ Thm> A sequence $\{a_n\}$ can have at most one limit.

pf) Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ $(L \ne M)$.

For $\varepsilon = \frac{|L-M|}{2} > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$n > N_1 \Rightarrow |a_n - L| < \frac{|L-M|}{2}$, $n > N_2 \Rightarrow |a_n - M| < \frac{|L-M|}{2}$

Take $N := \max\{N_1, N_2\}$ then $n > N \Rightarrow |L - M| = |L - a_n + a_n - M|$

$\le |a_n - L| + |a_n - M| < \frac{|L-M|}{2} + \frac{|L-M|}{2} = |L - M|$ and it is a contradiction. $\blacksquare$

Alternative proof : $\forall \varepsilon > 0$, $|L-M| \le |a_n - L| + |a_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore L = M$ ■

◎ Thm> If $\{a_n\}$ is a convergent sequence and $a_n \ge 0$ for $\forall n \ge n_0$ for some $n_0 \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = L \ge 0$.

pf) Suppose $\lim_{n\to\infty} a_n = L < 0$. For $\varepsilon = -\frac{L}{2} > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow$ $|a_n - L| < -\frac{L}{2} \Rightarrow L + \frac{L}{2} < a_n < L - \frac{L}{2} = \frac{L}{2} < 0$ and it is a contradiction. ■

◎ Thm> A real function f is continuous at a $\Leftrightarrow$ $\lim_{x\to a} f(x) = f(a)$ $\Leftrightarrow$ For every sequence $\{a_n\}$ that converges to a, $\lim_{n\to\infty} f(a_n) = f(a)$. $(a_n \ne a)$

pf) [$\Rightarrow$] Let $\lim_{n\to\infty} a_n = a$ and $\varepsilon > 0$ be given. Since f is continuous at a, $\exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. For $\delta > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - a| < \delta \Rightarrow |f(a_n) - f(a)| < \varepsilon$, $\therefore \lim_{n\to\infty} f(a_n) = f(a)$ ■

[$\Leftarrow$] Assume that f is not continuous at a, then $\exists \varepsilon_0 > 0$, $\forall \delta > 0$, $\exists x_\delta$ s.t. $|x_\delta - a| < \delta$ but $|f(x_\delta) - f(a)| \ge \varepsilon_0$. Thus,

For $\delta = 1$, $\exists x_1$ s.t. $|x_1 - a| < 1$, $|f(x_1) - f(a)| \ge \varepsilon_0$

For $\delta = \frac{1}{2}$, $\exists x_2$ s.t. $|x_2 - a| < \frac{1}{2}$, $|f(x_2) - f(a)| \ge \varepsilon_0$

$\vdots$

For $\delta = \frac{1}{n}$, $\exists x_n$ s.t. $|x_n - a| < \frac{1}{n}$, $|f(x_n) - f(a)| \ge \varepsilon_0$

Consider $\{x_n\}$ and $\{f(x_n)\}$. Since $\lim_{n\to\infty} |x_n - a| = 0$, $\lim_{n\to\infty} (x_n - a) = 0$ and $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) \ne f(a)$ and it is a contradiction. ■

◎ Note

$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right) = f(a)$, $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a)$.

ex) $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0$ ($\because f(x) = \sin x$ is continuous at 0 and $\lim_{n\to\infty} \frac{\pi}{n} = 0$.)

sukita1729@gmail.com

◎ Thm> If $b_n \le a_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.

pf) Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.
$n > N_1 \Rightarrow |b_n - L| < \varepsilon$, $n > N_2 \Rightarrow |c_n - L| < \varepsilon$
Take $N := \max\{N_1, N_2, n_0\}$, then $n > N \Rightarrow L - \varepsilon < b_n \le a_n \le c_n < L + \varepsilon$
$\Rightarrow |a_n - L| < \varepsilon$, $\lim_{n\to\infty} a_n = L$ ■

◎ Defn> * A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for $\forall n \ge 1$.
* A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for $\forall n \ge 1$.
* A sequence $\{a_n\}$ is called monotonic if it is either increasing or decreasing.

<Example> $a_n = \frac{n}{n^2+1}$, $f(x) = \frac{x}{x^2+1}$. Since $f(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$
and f is continuous on the interval $[1, \infty)$, f is decreasing on $[1, \infty)$.
$\Rightarrow f(n) = a_n > a_{n+1} = f(n+1)$ for $\forall n \in \mathbb{N}$, $\{a_n\}$ is decreasing.

◎ Thm> * A sequence $\{a_n\}$ is bounded above $\Leftrightarrow$ $\exists M \in \mathbb{R}$ s.t. $a_n \le M$ for $\forall n \in \mathbb{N}$
* A sequence $\{a_n\}$ is bounded above $\Leftrightarrow$ $\exists M \in \mathbb{R}$ s.t. $M \le a_n$ for $\forall n \in \mathbb{N}$
* A sequence $\{a_n\}$ is bounded $\Leftrightarrow$ $\{a_n\}$ is bounded below and above

◎ Defn> Let $\varnothing \ne S \subseteq \mathbb{R}$.
(a) The set S is said to be bounded above if $\exists u \in \mathbb{R}$ s.t. $s \le u$ for $\forall s \in S$.
Each such number u is called an upper bound for S.
(b) The set S is said to be bounded below if $\exists v \in \mathbb{R}$ s.t. $s \ge v$ for $\forall s \in S$.
Each such number v is called a lower bound for S.
(c) A set is said to be bounded if it is both bounded above and bounded below.
(d) A set is said to be unbounded if it is not bounded.

◎ Thm> Every convergent sequence is bounded.

pf) Let $\lim_{n\to\infty} a_n = L$. For $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - L| < 1$
$\Rightarrow |a_n| = |a_n - L + L| \le |a_n - L| + |L| < 1 + |L|$. Take
$M := \max\{|a_1|, |a_2|, \cdots, |a_N|, 1 + |L|\} \Rightarrow \forall n \in \mathbb{N}$, $|a_n| \le M$ and therefore
$\{a_n\}$ is bounded. ■

◎ Thm> If $\lim_{n\to\infty} a_n = L$ and $a_n \le \beta$ for $n \ge n_0$, then $L \le \beta$.

pf) Suppose $L > \beta$. For $\varepsilon_0 = L - \beta > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow$
$|a_n - L| < \varepsilon_0 = L - \beta \Rightarrow \beta = L - \varepsilon_0 < a_n < L + \varepsilon_0 = 2L - \beta \Rightarrow \beta < a_n$.
Take $N := \max\{n_0, N_1\}$ then $\beta < a_n \le \beta$ and it is a contradiction. ■

◎ Thm> If $\lim_{n\to\infty} a_n = L$ and $\alpha \le a_n \le \beta$ for $n \ge n_0$, then $\alpha \le L \le \beta$.

◎ Defn> Let $\varnothing \ne S \subseteq \mathbb{R}$.

* An upper bound u_0 for S is the least upper bound for $S \Leftrightarrow u_0 \le u$ for every upper bound u for $S \Leftrightarrow u_0 = \sup S \Leftrightarrow u_0 = \mathrm{lub} S$

* A lower bound v_0 for S is the greatest lower bound for $S \Leftrightarrow v_0 \ge v$ for every lower bound v for $S \Leftrightarrow v_0 = \inf S \Leftrightarrow v_0 = \mathrm{glb} S$

◎ The Completeness Axiom

Every nonempty subset of $\mathbb{R}$ that is bounded above(below) has a least upper bound(greatest lower bound) in $\mathbb{R}$.

◎ Monotone Sequence Theorem (MST) : Every bounded, monotonic sequence converges.
$\Leftrightarrow$ $\{a_n\}$: increasing & bounded above or decreasing & bounded below $\Rightarrow$ converges

pf) Suppose $\{a_n\}$ is increasing and bounded above. Let $S = \{a_n \mid n \ge 1\} \Rightarrow S$ has a least upper bound L by the Completeness Axiom. Let $\varepsilon > 0$ be given then $L - \varepsilon$ is not an upper bound for S, and $\exists a_N$ s.t. $L - \varepsilon < a_N \le L$.

Since $\{a_n\}$ is increasing, $n > N \Rightarrow L - \varepsilon < a_N \le a_n \le L < L + \varepsilon \Rightarrow |a_n - L| < \varepsilon$
$\therefore \lim_{n\to\infty} a_n = L = \sup S$ ■

<Example> $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$. Find the limit of $\{a_n\}$.

① $\{a_n\}$ is increasing since $a_1 = \sqrt{2} < \sqrt{2\sqrt{2}} = a_2$ and if $a_k < a_{k+1}$ then
$a_{k+1} = \sqrt{2a_k} < \sqrt{2a_{k+1}} = a_{k+2}$. Therefore, $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$.

② $\{a_n\}$ is bounded below by $\sqrt{2}$ and above by 2 since $\sqrt{2} \le a_1 = \sqrt{2} \le 2$ and if
$\sqrt{2} \le a_k \le 2$ then $\sqrt{2} \le \sqrt{2\sqrt{2}} \le a_{k+1} = \sqrt{2a_k} \le 2$.

Therefore, $\sqrt{2} \le a_n \le 2$ for $\forall n \in \mathbb{N}$.

It follows from the MST that $\lim_{n\to\infty} a_n$ exists. Let $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n}$ and $L = \sqrt{2L}$, $L = 0$ or $L = 2$. Since $\sqrt{2} \le a_n \le 2$, $\sqrt{2} \le L \le 2$ and $L = 2$. $\therefore \lim_{n\to\infty} a_n = 2$ ■

<Example> $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$. Find the limit of $\{a_n\}$.

① $\{a_n\}$ is increasing since $a_1 = 2 < 4 = a_2$ and if $a_k < a_{k+1}$ then $a_{k+1} = \frac{1}{2}(a_k + 6) < \frac{1}{2}(a_{k+1} + 6) = a_{k+2}$. Therefore, $a_n < a_{n+1}$ for $\forall n \in \mathbb{N}$.

② $\{a_n\}$ is bounded above by 6 since $a_1 = 2 \le 6$ and if $a_k \le 6$ then $a_{k+1} = \frac{1}{2}(a_k + 6) \le 6$. Therefore, $a_n \le 6$ for $\forall n \in \mathbb{N}$.

By the MST, $\lim_{n\to\infty} a_n$ exists. Let $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}(a_n + 6)\right)$ and $L = \frac{1}{2}(L+6)$, $L = 6$. $\therefore \lim_{n\to\infty} a_n = 6$ ■

<Example> Let $e_n = \left(1 + \frac{1}{n}\right)^n$. Find the limit of $\{e_n\}$.

① $e_n = \left(1+\frac{1}{n}\right)^n = 1 + {}_n\mathrm{C}_1\left(\frac{1}{n}\right) + {}_n\mathrm{C}_2\left(\frac{1}{n}\right)^2 + {}_n\mathrm{C}_3\left(\frac{1}{n}\right)^3 + \cdots + {}_n\mathrm{C}_n\left(\frac{1}{n}\right)^n$

$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!} \cdot \left(\frac{1}{n}\right)^n$

$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$,

$e_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1 + {}_{n+1}\mathrm{C}_1\left(\frac{1}{n+1}\right) + {}_{n+1}\mathrm{C}_2\left(\frac{1}{n+1}\right)^2 + {}_{n+1}\mathrm{C}_3\left(\frac{1}{n+1}\right)^3 + \cdots + {}_{n+1}\mathrm{C}_{n+1}\left(\frac{1}{n+1}\right)^{n+1}$

$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$

Therefore, $e_n < e_{n+1}$ for $\forall n \in \mathbb{N}$.

② $e_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+\frac{1}{1-1/2} = 3.$

Therefore, $e_n \leq 3$ for $\forall n \in \mathbb{N}$.

By the MST, $\lim_{n\to\infty} e_n$ exists, and it is actually the so-called Euler's constant e. ■

<Example> $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots = \lim_{n\to\infty} e_n$.

pf) $S_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 3$, $S_n < S_{n+1}$.

$\Rightarrow$ By the MST, $\lim_{n\to\infty} S_n = L = \sum_{n=0}^{\infty} \frac{1}{n!}$. Let $S_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $T_n = \left(1+\frac{1}{n}\right)^n$.

① $T_n = \left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} = S_n$. Therefore, $T_n \leq S_n$ for $\forall n \in \mathbb{N}$.

$e = \lim_{n\to\infty} T_n \leq \lim_{n\to\infty} S_n = L$.

② $T_n \geq \left(1+\frac{1}{n}\right)^m = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$ $(n \geq m)$

Fix the value of m and send $n\to\infty$, then $e = \lim_{n\to\infty} T_n \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!} = S_m$.

Therefore, $\forall m \in \mathbb{N}$, $S_m \leq e$ and $L = \lim_{n\to\infty} S_n \leq e$.

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = L = \sum_{n=0}^{\infty} \frac{1}{n!}$ ■

<Example> Prove $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$

<Example> For sequence $\{x_n\}$ such that $x_n > 0$, let $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha$. Prove :

① $0 \leq \alpha < 1$: $\lim_{n\to\infty} x_n = 0$

② $\alpha > 1$: $\lim_{n\to\infty} x_n = \infty$

③ $\alpha = 1$: Nothing

§ 11.2 Series

◎ (Infinite) Series

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n \text{ or } \sum a_n$$

◎ Defn>

Given a series $\sum a_n$, let S_n denote its nth partial sum : $S_n = \sum_{k=1}^{n} a_k$. If $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n \in \mathbb{R}$, then $\sum a_n$ is called convergent. Then

$a_1 + a_2 + a_3 + \cdots = S$ or $\sum_{n=1}^{\infty} a_n = S$. S is called the sum of the series $\sum a_n$.

If $\{S_n\}$ is divergent then $\sum a_n$ is divergent.

∗ Note : $\sum_{n=1}^{\infty} a_n = S = \lim_{n\to\infty} S_n$

<Example> Geometric Series

If $|r| < 1$ then the series $a + ar + ar^2 + \cdots$ converges to $\frac{a}{1-r}$ $(a \neq 0)$.

Therefore, $\sum ar^{n-1}$ is convergent $\Leftrightarrow$ $|r| < 1$ $(a \neq 0)$.

<Example> Harmonic Series

$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum \frac{1}{n} = \infty$, since $S_{2^n} > 1 + \frac{n}{2}$ and therefore $\lim_{n\to\infty} S_n = \infty$.

◎ Thm> $\sum a_n$ is convergent $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$

pf) Let $\lim_{n\to\infty} S_n = S$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0. \blacksquare$$

◎ Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

<Example> For the sequence $a_n = \frac{n^2}{5n^2+4}$, $\sum_{n=1}^{\infty} a_n$ is divergent since

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0.$$

<Example> For the sequence $a_n = n\sin\frac{1}{n}$, $\sum_{n=1}^{\infty} a_n$ is divergent since

$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{t\to 0+}\frac{\sin t}{t} = 1 \neq 0.$

◎ Thm> If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are convergent, then for constant $c\in\mathbb{R}$,

① $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

② $\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

③ $\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$

proof by the definition of $\sum_{n=1}^{\infty} a_n$ and the limit laws of a converging limit.

◎ Note

A finite number of terms does not affect the convergence or divergence of a series.

<Example> $\sum_{n=2}^{\infty}\frac{1}{n^3-n} = \sum_{n=2}^{\infty}\frac{1}{n(n-1)(n+1)} = \frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{1}{n(n-1)} - \frac{1}{n(n+1)}\right]$

$= \lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{2} - \frac{1}{n(n+1)}\right) = \frac{1}{4}.$

<Example> $\sum_{n=1}^{\infty}\tan^{-1}n$ is divergent since $\lim_{n\to\infty}\tan^{-1}n = \frac{\pi}{2}\neq 0.$

<Example> $\sum_{n=1}^{\infty}\sin n$ is convergent since $\lim_{n\to\infty}\sin n$ is not convergent.

<Example> $\sum_{n=1}^{\infty}(x+2)^n$ is convergent if $-3 < x < -1.$

<Example> $\sum_{n=0}^{\infty}e^{nx}$ is convergent if $x<0.$

<Example> $\sum_{n=0}^{\infty}\frac{\sin^n x}{3^n}$ is convergent regardless of x if $x\in\mathbb{R}.$

<Example> $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \left[\frac{1}{n!} - \frac{1}{(n+1)!} \right] = \lim_{n \to \infty} \left(1 - \frac{1}{(n+1)!} \right) = 1$

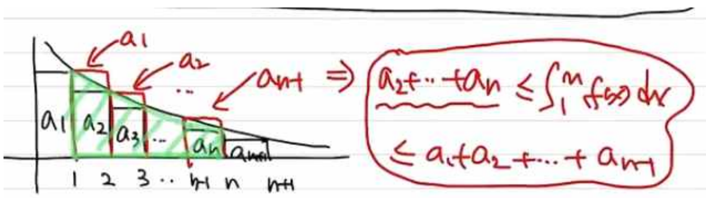
§ 11.3 The Integral Test and Estimate of Sums

(Q) The convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$?

◎ The Integral Test

Suppose f is a continuous, positive, and a decreasing function on $[1, \infty)$ and $f(n) = a_n$ for $\forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow$ $\int_1^{\infty} f(x)dx$ is convergent.

pf)



If $\int_1^{\infty} f(x)dx$ is convergent then $a_2 + a_3 + \cdots + a_n = S_n - a_1 \le \int_1^n f(x)dx \le \int_1^{\infty} f(x)dx$ and $S_n \le a_1 + \int_1^{\infty} f(x)dx = M$: An upper bound. Therefore, S_n is increasing and $S_n \le M$ for $M \in \mathbb{R}$. By the MCT, $\lim_{n \to \infty} S_n$ exists and $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^{\infty} f(x)dx$ is divergent then $\lim_{n \to \infty} \int_1^n f(x)dx = \infty$. Since $\int_1^n f(x)dx \le a_1 + \cdots + a_{n-1} = S_{n-1}$, $\lim_{n \to \infty} S_{n-1} = \infty$ and $\lim_{n \to \infty} S_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ is divergent.

<Example> For $\sum_{n=1}^{\infty} \frac{\ln n}{n}$, let $f(x) = \frac{\ln x}{x}$ then $f(x) > 0$ for $x > 1$, and $f'(x) < 0$ for $x > e$. Therefore, f is continuous, positive, and decreasing on $[e, \infty)$.

Since $\int_e^{\infty} \frac{\ln x}{x} dx = \lim_{b \to \infty} \int_e^b \frac{\ln x}{x} dx = \lim_{b \to \infty} \left[\frac{1}{2} (\ln x)^2 \right]_e^b = \lim_{b \to \infty} \frac{1}{2} ((\ln b)^2 - 1) = \infty$.

Since $\int_e^{\infty} f(x)dx$ is divergent, $\int_1^{\infty} f(x)dx$ is also divergent and therefore $\sum_{n=1}^{\infty} a_n$ is divergent.

<Example> p-series : convergence of $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$

sol) (i) $p>1$: $f(x)=\frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$.

$\int_1^{\infty}\frac{1}{x^p}dx=\int_1^{\infty}x^{-p}dx=\lim_{t\to\infty}\int_1^t x^{-p}dx=\lim_{t\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^t$

$=\lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1)=\lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right)=\frac{1}{p-1}$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

(ii) $0<p<1$: $f(x)=\frac{1}{x^p}$ is continuous, positive, and decreasing on $[1,\infty)$.

$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{t\to\infty}\frac{1}{1-p}(t^{1-p}-1)=\infty$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

(iii) $p=1$: $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges since it is the harmonic series.

(iv) $p\le 0$: $\lim_{n\to\infty}\frac{1}{n^p}=\lim_{n\to\infty}n^{-p}=\infty$, $\lim_{n\to\infty}\frac{1}{n^p}=\lim_{n\to\infty}1=1\ne 0$ if $p=0$

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

$\Rightarrow$ $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$.

<Example> Determine the convergence of $\sum_{n=1}^{\infty}ne^{-n^2}$.

sol) Let $f(x)=\frac{x}{e^{x^2}}$ then $f'(x)<0$ for $x\ge 1$ and f is continuous, positive, and decreasing on $[1,\infty)$. Now we should check the convergence of $\int_1^{\infty}xe^{-x^2}dx$. Since $\int_1^{\infty}xe^{-x^2}dx=\frac{1}{2}\int_1^{\infty}\frac{1}{e^t}dt=\lim_{b\to\infty}\frac{1}{2}\int_1^b e^{-t}dt$ $(t=x^2)$

$=\lim_{b\to\infty}\left[-\frac{1}{2}e^{-t}\right]_1^b=\lim_{b\to\infty}\left(-\frac{1}{2e^b}+\frac{1}{2e}\right)=\frac{1}{2e}$, $\sum_{n=1}^{\infty}ne^{-n^2}$ is convergent.

<Example> Determine the convergence of $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$.

sol) Let $f(x)=\frac{1}{2^{\ln x}}$ then $f'(x)<0$ for $x\ge 1$ and f is continuous, positive, and decreasing on $[1,\infty)$. Now we should check the convergence of

$\int_1^\infty \frac{1}{2^{\ln x}}dx$. Since $\int_1^\infty \frac{1}{2^{\ln x}}dx = \int_0^\infty \left(\frac{e}{2}\right)^u du \ (u = \ln x)$

$= \lim_{b\to\infty}\left[\frac{(e/2)^u}{\ln(e/2)}\right]_0^b = \lim_{b\to\infty}\frac{1}{\ln(e/2)}\left(\left(\frac{e}{2}\right)^b - 1\right) = \infty$, $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$ is divergent.

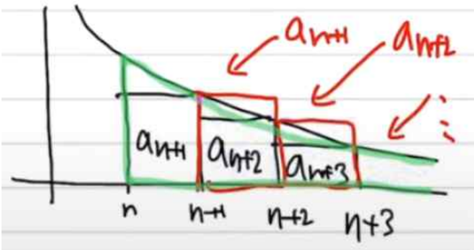
◎ Estimating the Sum of a Series

$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = \lim_{n\to\infty} S_n$, let

$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$ then $S = S_n + R_n$.

◎ Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$ for $\forall k \in \mathbb{N}$ where f is a continuous, positive, and decreasing function on $[n, \infty)$. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_n^\infty f(x)dx$.

pf) 

<Example> $\sum_{n=1}^{\infty}\frac{1}{n^3}$

Since $\int_n^\infty \frac{1}{x^3}dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$, if $n \ge 32$ then

$R_n \le \frac{1}{2n^2} \le 0.0005$,

◎ Note

$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_n^\infty f(x)dx \Rightarrow$

$S_n + \int_{n+1}^{\infty} f(x)dx \le S = R_n + S_n \le S_n + \int_n^\infty f(x)dx$

<Example> $\sum_{n=1}^{\infty}\frac{1}{n^2+4}$ is convergent since $\int_1^\infty \frac{1}{x^2+4}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+4}dx$

$= \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_1^t = \lim_{t\to\infty}\frac{1}{2}\left(\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right)$.

<Example> $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+1}$

Let $f(x)=\dfrac{x}{x^4+1}$ then $f'(x)<0$ for $x\ge 1$ and f is continuous, positive, and decreasing on $[1,\infty)$. Since $\displaystyle\int_1^{\infty}\frac{x}{x^4+1}dx=\lim_{t\to\infty}\int_1^t\frac{x}{x^4+1}dx=\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(t^2)\right]_1^t$

$\displaystyle=\lim_{t\to\infty}\frac{1}{2}\left(\tan^{-1}(t^2)-\tan^{-1}(1)\right)=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{8}$, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+1}$ is convergent.

§ 11.4 The Comparison Test

◎ The Comparison Test (CT)

Suppose $\displaystyle\sum_{n=1}^{\infty}a_n$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ are series with positive terms.

(i) If $\displaystyle\sum_{n=1}^{\infty}b_n$ is convergent and $a_n\le b_n$ for $\forall n\in\mathbb{N}$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ is convergent.

(ii) If $\displaystyle\sum_{n=1}^{\infty}b_n$ is divergent and $a_n\ge b_n$ for $\forall n\in\mathbb{N}$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ is divergent.

pf) Let $\displaystyle S_n=\sum_{i=1}^{n}a_i$, $\displaystyle T_n=\sum_{i=1}^{n}b_i$.

(i) $a_n, b_n>0 \Rightarrow \{S_n\}, \{T_n\}$ are increasing

Since $\displaystyle\sum_{n=1}^{\infty}b_n$ is convergent, $\{S_n\}$ is bounded above from the fact that

$\displaystyle S_n=a_1+\cdots+a_n\le b_1+\cdots+b_n=T_n\le\sum_{n=1}^{\infty}b_n.$

Since $\{S_n\}$ is increasing, $\displaystyle\lim_{n\to\infty}S_n=\sum_{n=1}^{\infty}a_n$ exists and $\displaystyle\sum_{n=1}^{\infty}a_n$ is convergent by MCT. ■

(ii) Since $b_n>0$ for $\forall n\in\mathbb{N}$, $\displaystyle\sum_{n=1}^{\infty}b_n=\lim_{n\to\infty}T_n=\infty$. Since

$T_n=b_1+\cdots+b_n\le a_1+\cdots+a_n=S_n$, $\displaystyle\lim_{n\to\infty}S_n=\infty$ and $\displaystyle\sum_{n=1}^{\infty}a_n$ is divergent. ■

<Example> Determine the convergence of $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k}$.

sol) Since $\dfrac{\ln k}{k}>\dfrac{1}{k}$ for $k\ge 3$ and $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k}=\infty$, $\displaystyle\sum_{k=3}^{\infty}\frac{\ln k}{k}$ is divergent and therefore

$\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is divergent.

<Example> $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent since

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

<Example> $\sum_{n=1}^{\infty} \frac{5}{3n-2}$ is divergent since $\frac{5}{3n-2} > \frac{5}{3n} > \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

<Example> $\sum_{n=1}^{\infty} \frac{\ln n}{2n^3-1}$ is convergent since $\frac{\ln n}{2n^3-1} \le \frac{n}{2n^3-1} \le \frac{n}{n^3} = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

◎ The Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges or diverges.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent.

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} b_n$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is divergent.

pf) (i) Since $\frac{c}{2} > 0$, $\exists N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}$, $\frac{c}{2} < \frac{a_n}{b_n} < \frac{3}{2}c$, $\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$. If $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=N}^{\infty} a_n$ is convergent by the CT, and therefore $\sum_{n=1}^{\infty} a_n$ is convergent. If $\sum_{n=1}^{\infty} b_n$ is divergent, $\sum_{n=N}^{\infty} a_n$ is divergent by the CT, and therefore $\sum_{n=1}^{\infty} a_n$ is divergent. ■

(ii) $\exists N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow \left| \frac{a_n}{b_n} \right| = \frac{a_n}{b_n} < 1$, $a_n < b_n$. Since $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=N}^{\infty} a_n$ is convergent by the CT, and therefore $\sum_{n=1}^{\infty} a_n$ is convergent. ■

(iii) $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \left|\frac{a_n}{b_n}\right| = \frac{a_n}{b_n} > 1$, $a_n > b_n > 0$. Since $\sum_{n=1}^{\infty} b_n$ is divergent, $\sum_{n=N}^{\infty} a_n$ is divergent by the CT, and therefore $\sum_{n=1}^{\infty} a_n$ is divergent. ■

<Example> Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$.

sol) Let $a_n = \frac{1}{\sqrt{n+1}}$, $b_n = \frac{1}{\sqrt{n}}$, then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by the p-series test. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 > 0$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

<Example> Determine the convergence of $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.

sol) Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = \frac{2}{\sqrt{n}}$ then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ is divergent by the p-series test. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} = 1 > 0$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is divergent.

<Example> Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^3+100}$.

sol) Let $a_n = \frac{1}{n^3+100}$, $b_n = \frac{1}{n^3}$ then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p-series test. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3+100} = 1 > 0$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^3+100}$ is convergent.

<Example> Determine the convergence of $\sum_{n=1}^{\infty} \frac{n\ln n+1}{n^2+5}$.

sol) Let $a_n = \frac{n\ln n+1}{n^2+5}$, $b_n = \frac{1}{n}$ then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the p-series test. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2\ln n+n}{n^2+5} = \infty$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n\ln n+1}{n^2+5}$ is divergent.

§ 11.5 Alternating Series and Absolute Convergence

◎ Thm> Alternating Series Test (AST)

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \quad (b_n > 0)$ satisfies the conditions (i) $b_{n+1} \le b_n$ for $\forall n \in \mathbb{N}$ and (ii) $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent.

pf) Let $S_{2n} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2n-1} - b_{2n} = S_{2n-2} + b_{2n-1} - b_{2n}$ $\ge S_{2n-2} \Rightarrow \{S_{2n}\}$ increases. Since
$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - b_{2n} \le b_1$, $\{S_{2n}\}$ is bounded above. By the MCT, $\{S_{2n}\}$ is convergent and let $\lim_{n\to\infty} S_{2n} = S$. Since $S_{2n+1} = S_{2n} + b_{2n+1}$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{2n+1} = 0$, $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = S$ and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent. ■

<Example> $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$

Let $f(x) = \frac{x^2}{x^3+1}$ then $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$ and f is decreasing on $[2, \infty)$.

Since $\lim_{n\to\infty}\frac{n^2}{n^3+1} = 0$, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ is convergent by the AST.

◎ Alternating Series Estimation Theorem

If $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is an alternating series, then $|R_n| = |S - S_n| \le b_{n+1}$.

pf) Since $S_{2n} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2n-1} - b_{2n} = S_{2n-2} + b_{2n-1} - b_{2n}$,
$S_{2n} \ge S_{2n-1}$ and $\{S_{2n}\}$ increases. Since $S_{2n+1} = S_{2n-1} - (b_{2n} - b_{2n+1}) \le S_{2n-1}$,
$\{S_{2n-1}\}$ decreases and $S_2 \le S_4 \le \cdots \le S \le \cdots \le S_3 \le S_1$.
$\Rightarrow \forall n, k \in \mathbb{N}$, $S_{2k} \le S \le S_{2n+1}$. If $k = n$ then $S_{2n} \le S \le S_{2n+1}$ and
$0 \le S - S_{2n} \le b_{2n+1}$, $|S - S_{2n}| \le b_{2n+1}$. If $k = n+1$ then $S_{2n+2} \le S \le S_{2n+1}$ and
$-b_{2n+2} \le S - S_{2n+1} \le 0$, $|S - S_{2n+1}| \le b_{2n+2}$. Thus, $|R_n| = |S - S_n| \le b_{n+1}$. ■

<Example> $S = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$, $|S - S_6| \le \frac{1}{7!}$.

◎ Absolute Convergence and Conditional Convergence

* $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} |a_n|$ is convergent

* $\sum_{n=1}^{\infty} a_n$ is conditionally convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent

◎ Thm> $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent

pf) $\forall n \in \mathbb{N}$, $0 \le a_n + |a_n| \le 2|a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, $\sum_{n=1}^{\infty} 2|a_n|$ is convergent. By the CT, $\sum_{n=1}^{\infty} (a_n + |a_n|)$ is convergent. Therefore,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} [(a_n + |a_n|) - |a_n|]$ is convergent. ■

<Example> $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent ($\because p = 2 > 1$) and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

<Example> $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1} + \frac{\cos 2}{2} + \frac{\cos 3}{3} + \cdots$

Since $0 \le \left| \frac{\cos n}{n^2} \right| \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ and $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent by the CT.

<Example> (1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ (2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ (3) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$

(1) Since $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is absolutely convergent and it converges.

(2) $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, $b_{n+1} = \frac{1}{\sqrt[3]{n+1}} < b_n = \frac{1}{\sqrt[3]{n}}$.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is convergent by the AST.

(3) Since $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \ne 0$, $\lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$ does not exist and therefore

$\sum_{n=1}^{\infty}(-1)^n\frac{n}{2n+1}$ is divergent.

◎ Defn> A series $\sum_{n=1}^{\infty}b_n$ is a rearrangement of a series $\sum_{n=1}^{\infty}a_n \Leftrightarrow$ There exists a bijection of $f:\mathbb{N}\to\mathbb{N}$ s.t. $b_n = a_{f(n)}$.

◎ Thm> If $\sum_{n=1}^{\infty}a_n$ is absolutely convergent and $\sum_{n=1}^{\infty}b_n$ is any rearrangement of $\sum_{n=1}^{\infty}a_n$, then $\sum_{n=1}^{\infty}b_n$ is absolutely convergent where $\sum_{n=1}^{\infty}b_n = \sum_{n=1}^{\infty}a_n$.

◎ Thm> If $\sum_{n=1}^{\infty}a_n$ is conditionally convergent and $r\in\mathbb{R}$, there exists a rearrangement $\sum_{n=1}^{\infty}b_n$ of $\sum_{n=1}^{\infty}a_n$ such that $\sum_{n=1}^{\infty}b_n = r$.

<Example> $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n-1}$ is divergent since $\lim_{n\to\infty}\frac{3n-1}{2n-1} = \frac{3}{2}$ and $\lim_{n\to\infty}(-1)^n\frac{3n-1}{2n-1}$ is divergent.

<Example> $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{1/n}}{n}$ is convergent since $\lim_{n\to\infty}\frac{1}{n}e^{1/n} = 0$ and if $f(x) = \frac{e^{1/x}}{x}$ then since $f'(x) < 0$ for $x \geq 1$, $\left\{\frac{e^{1/n}}{n}\right\}$ is decreasing.

<Example> $\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!}{(2^n n!)^2}$

<Example> $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n(\ln n)^2}$

$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ is convergent since $\int\frac{1}{x(\ln x)^2}dx = -\frac{1}{\ln x} + C$ and

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_2^b\frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_2^b = \lim_{b\to\infty}\left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Therefore, $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n(\ln n)^2}$ is absolutely convergent and it converges.

($f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2,\infty)$.)

<Example> $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ is absolutely convergent since $\left|\frac{(-1)^n}{n\sqrt{n}}\right| = \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

§ 11.6 The Ratio and Root Tests

◎ The (Limit) Ratio Test

For the series $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$,

① $0 \le \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $L = \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

③ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$ Inconclusive ($\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

pf) ① $0 \le \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$

$\exists\, r$ s.t. $L < r < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, for $\varepsilon = r - L > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$n \ge N \Rightarrow \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \varepsilon \Rightarrow L - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon = r \Rightarrow |a_{n+1}| < r|a_n|$.

Therefore, for $\forall k \in \mathbb{N}$, $|a_{N+k}| < r^k |a_N|$. Since $\sum_{k=1}^{\infty} r^k |a_N|$ is convergent, $\sum_{k=1}^{\infty} |a_{N+k}|$ is convergent by the CT. Since N is finite, $\sum_{n=1}^{\infty} |a_n|$ is convergent since $\sum_{n=N+1}^{\infty} |a_n|$ is convergent. ■

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $L = \infty$

$\exists\, r$ s.t. $1 < r < L$. For $\varepsilon = L - r$, $\exists\, N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow r = L - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$ $\Rightarrow r|a_n| < |a_{n+1}| \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > r > 1$. If $L = \infty$, then for $M = 1$, $\exists\, N \in \mathbb{N}$ s.t. $n \ge N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow |a_{n+1}| > |a_n|$. Since $\lim_{n\to\infty} |a_n| \neq 0$, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ is divergent. ■

<Example> $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3n^3} \to \frac{1}{3} < 1$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

<Example> $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$

Since $\left|\frac{b_{n+1}}{b_n}\right| = \left(1+\frac{1}{n}\right)^n \to e > 1$ as $n \to \infty$, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

◎ The (Limit) Root Test

For the series $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$,

① $0 \leq \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

② $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $L = \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent

③ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \Rightarrow$ Inconclusive ($\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

pf) ① $0 \leq \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$

$\exists r$ s.t. $L < r < 1$. Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, for $\varepsilon = r - L > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow$ $\left|\sqrt[n]{|a_n|} - L\right| < \varepsilon \Rightarrow \sqrt[n]{|a_n|} < L + \varepsilon = r \Rightarrow 0 \leq |a_n| < r^n$.

Since $\sum_{n=N}^{\infty} r^n$ is convergent, $\sum_{n=N}^{\infty} |a_n|$ is convergent by the CT. Thus, $\sum_{n=1}^{\infty} |a_n|$ is convergent. ■

② $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $L = \infty$

$\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} > 1 \Rightarrow |a_n| > 1 \Rightarrow \lim_{n\to\infty} |a_n| \neq 0$ and $\lim_{n\to\infty} a_n \neq 0$.

Therefore, $\sum_{n=1}^{\infty} a_n$ is divergent. ■

<Example> $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent since $\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$.

<Example> $\sum_{n=1}^{\infty} b_n$ where $b_n = \begin{cases} \dfrac{n}{2^n} & n \text{ is odd} \\ \dfrac{1}{2^n} & n \text{ is even} \end{cases}$ (Only the Root Test holds.)

<Example> $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{2^n n!}{(2n)!} = 1 - \dfrac{2!}{1 \cdot 3} + \dfrac{3!}{1 \cdot 3 \cdot 5} - \dfrac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$ (con)

<Example> $\sum_{n=1}^{\infty} \left(1 + \dfrac{1}{n}\right)^{n^2}$ (div)

<Example> $\sum_{n=1}^{\infty} \dfrac{\sin\left(\dfrac{n\pi}{6}\right)}{1 + n\sqrt{n}}$ (con)

<Example> $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{\sqrt{n}\ \ln n}$ (div)

<Example> For $b_n > 0$ and $\lim_{n \to \infty} b_n = \dfrac{1}{2}$, $\sum_{n=1}^{\infty} \dfrac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n}$ (div)

<Example> $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{1 + n^2}$ (con)

◎ The Ratio Test

For the series $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$,

① $\exists\, r \in (0, 1)$, $\exists\, k \in \mathbb{N}$ s.t. $\forall\, n \geq k$, $\left|\dfrac{a_{n+1}}{a_n}\right| \leq r \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

② $\exists\, r > 1$, $\exists\, k \in \mathbb{N}$ s.t. $\forall\, n \geq k$, $\left|\dfrac{a_{n+1}}{a_n}\right| \geq r \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

pf) ① $\forall\, n \geq k$, $\left|\dfrac{a_{n+1}}{a_n}\right| \leq r \Leftrightarrow |a_{n+1}| \leq r|a_n|$, $|a_{k+n}| \leq r^n |a_k|$ for $\forall\, n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} r^n |a_k|$ is convergent, $\sum_{n=1}^{\infty} |a_{k+n}|$ is convergent by the CT and $\sum_{n=1}^{\infty} a_n$ is convergent.

② $\forall\, n \geq k$, $\left|\dfrac{a_{n+1}}{a_n}\right| \geq r$, $r > 1 \Leftrightarrow |a_{n+1}| \geq r|a_n|$, $|a_{k+n}| \geq r^n |a_k|$ for $\forall\, n \in \mathbb{N}$.

Since $\lim_{n\to\infty} r^n |a_k| = \infty$, $\lim_{n\to\infty} |a_{k+n}| = \infty$ and $\lim_{n\to\infty} |a_n| = \infty$. Therefore, $\lim_{n\to\infty} a_n$ does not exist and $\sum_{n=1}^{\infty} a_n$ is divergent.

◎ The Root Test

For the series $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$,

① $\exists\, r \in (0, 1)$, $\exists\, k \in \mathbb{N}$ s.t. $\forall\, n \geq k$, $|a_n|^{\frac{1}{n}} \leq r \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

② $\exists\, r > 1$, $\exists\, k \in \mathbb{N}$ s.t. $\forall\, n \geq k$, $|a_n|^{\frac{1}{n}} \geq r \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

pf) The proof of the convergence of a series $\sum_{n=1}^{\infty} a_n$ is an application of the comparison test. If for all $n \geq N \in \mathbb{N}$, $|a_n|^{\frac{1}{n}} \leq k < 1$, then $|a_n| \leq k^n < 1$. Since the geometric series $\sum_{n=N}^{\infty} k^n$ converges, do does $\sum_{n=N}^{\infty} |a_n|$ by the comparison test. Hence $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $|a_n|^{\frac{1}{n}} > 1$ for infinitely many n, then a_n fails to converge to 0, hence the series is divergent. ■

§ 11.8 Power Series

◎ A power series is of the form $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ for $c_n \in \mathbb{R}$. Then the domain of the function f is $D(f) = \left\{ x \in \mathbb{R} \,\middle|\, \sum_{n=0}^{\infty} c_n x^n \text{ converges} \right\}$.

◎ In general, $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$ is called a power series in $(x-a)$, a power series centered at a, or a power series about a.

<Example> When does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

sol) Let $a_n = \frac{(x-3)^n}{n}$ then $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} |x-3|$,

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-3| < 1$ and $2 < x < 4$. If $x = 2$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by AST. If $x = 4$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore, the interval of convergence is $[2, 4)$ and the radius of convergence is $R = 1$.

<Example> When does $\sum_{n=0}^{\infty} n!x^n$ converge?

sol) Let $a_n = n!x^n$ then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ for all $x \neq 0$. Therefore, $I = \{0\}$ and $R = 0$.

<Example> When does $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

sol) Let $a_n = \frac{x^n}{(2n)!}$ then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(2n+2)!} \times \frac{(2n)!}{x^n}\right| = \frac{|x|}{(2n+2)(2n+1)}$ $\to 0 < 1$ for $\forall x \in \mathbb{R}$. Therefore, $I = \mathbb{R}$ and $R = \infty$.

<Example> When does $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ converge?

sol) Let $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{2^{2n+2}(n+1)!(n+1)!} \times \frac{2^{2n}n!n!}{x^{2n}}\right|$ $= \frac{x^2}{(n+1)^2} \to 0 < 1$. Therefore, $I = \mathbb{R}$ and $R = \infty$.

◎ Thm> For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities :

(i) The series converges only when $x = a$. $(R = 0)$
(ii) The series converges for $\forall x \in \mathbb{R}$. $(R = \infty)$
(iii) There is a positive number R such that the series converges if $|x-a| < R$ and the series diverges if $|x-a| > R$.

<Example> When does $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ converge?

sol) Let $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$ then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \times \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$ $= \frac{3\sqrt{n+1}}{\sqrt{n+2}}|x| \to 3|x| < 1$ if $|x| < \frac{1}{3}$. If $x = \frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent by AST and if $x = -\frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent. Therefore, $I = \left(-\frac{1}{3}, \frac{1}{3}\right]$ and $R = \frac{1}{3}$.

<Example> When does $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ converge?

sol) Let $a_n = \frac{n(x+2)^n}{3^{n+1}}$ then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \times \frac{3^{n+1}}{n(x+2)^n}\right|$ $= \frac{n+1}{3n}|x+2| \rightarrow \frac{1}{3}|x+2| < 1$ if $|x+2| < 3$. If $x = -5$, $\sum_{n=0}^{\infty} \frac{(-1)^n n}{3}$ is divergent and if $x = 1$, $\sum_{n=0}^{\infty} \frac{n}{3}$ is divergent. Therefore, $I = (-5, 1)$ and $R = 3$.

<Example> Find the interval of convergence and radius of convergence for series $\sum_{n=2}^{\infty} \frac{5^n}{n} x^n$, $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$, $\sum_{n=1}^{\infty} 2^n n^2 x^n$, $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

(Ans) $R_1 = \frac{1}{5}$, $I_1 = \left[-\frac{1}{5}, \frac{1}{5}\right)$, $R_2 = 4$, $I_2 = (-4, 4]$, $R_3 = \frac{1}{2}$, $I_3 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $R_4 = 2$, $I_4 = [-4, 0)$

§ 11.9 Representations of Functions as Power Series

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ $(|x| < 1)$: A power series representation of $\frac{1}{1-x}$ on the interval $(-1, 1)$

<Example> $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x| < 1$

<Example> $\frac{x^3}{x+2} = \frac{x^3}{2} \times \frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{x^3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$, $|x| < 2$

◎ Differentiation and Integration of Power Series

◎ Thm> If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$ is differentiable on $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ and $R' = R$.

(ii) $\int f(x)dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots$

$= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ and $R' = R$.

<Example> Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and $a_n = \frac{x^n}{n^2}$.

Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \times \frac{n^2}{x^n}\right| = \frac{n^2}{(n+1)^2}|x| \to |x| < 1$ when $|x| < 1$,

$R = 1$. When $x = \pm 1$, $\sum_{n=1}^{\infty}\left|\frac{(\pm 1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent and $I = [-1, 1]$.

Also, $f'(x) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$ and let $b_n = \frac{x^{n-1}}{n}$ then

$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^n}{n+1} \times \frac{n}{x^{n-1}}\right| = \frac{n}{n+1}|x| \to |x| < 1$ when $|x| < 1$, $R = 1$.

When $x = 1$, $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent but when $x = -1$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent by AST. Therefore, $I = [-1, 1)$.

<Example> Derive the power series expression for $\frac{1}{(1-x)^2}$.

sol) Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ $(|x| < 1)$,

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$ $(|x| < 1)$.

<Example> $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ $(|x| < 1)$. By integrating both sides, we obtain $\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, $|x| < 1$. When $x = 0$, $C = 0$.

Therefore, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ $(|x| < 1)$.

<Example> Derive the power series expression for $\tan^{-1}x$.

sol) Since $(\tan^{-1}x)' = \frac{1}{1+x^2}$, $\tan^{-1}x = \int \frac{1}{1+x^2}dx$

$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ and $C = 0$ when $x = 0$. Therefore,

$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$. Let $a_n = (-1)^n\frac{x^{2n+1}}{2n+1}$ then

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+3}}{2n+3} \times \frac{2n+1}{x^{2n+1}}\right| = \frac{2n+1}{2n+3}|x|^2 \rightarrow |x|^2 < 1$ when $|x| < 1$ and $R = 1$.

<Example> $\int \frac{1}{1+x^7}dx = \int \frac{1}{1-(-x^7)}dx = \int (1-x^7+x^{14}-x^{21}+\cdots)dx$

$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$

§ 11.10 Taylor and Mclaurin Series

$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \ |x-a| < R.$

Since $f(a) = c_0$, $f'(a) = c_1$, $f''(a) = 2!c_2$, $\cdots$, $f^{(n)}(a) = n!c_n$ and

$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and this is called the Taylor series of the function f at a, about a, or centered at a.

Especially when $a = 0$, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ is called the Mclaurin series of function f.

Not all functions can be expressed by power series, and some are even different from its Taylor series.

<Example> $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \ |x| < 1$

<Example> $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, let $a_n = \frac{x^n}{n!}$ then

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \rightarrow 0$ for $\forall x \neq 0$, and the series converges when $x = 0$. Therefore, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for $\forall x \in \mathbb{R}$ and $R = \infty$.

(Q) Under what circumstances is a function equal to the sum of its Taylor series?

$\Leftrightarrow$ When can we say that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

sol) Define $T_n(x)$ as $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ which is the nth-degree Taylor polynomial of f at a. Now $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \lim_{n\to\infty} T_n(x)$. Let

$R_n(x) = f(x) - T_n(x)$, then $f(x) = T_n(x) + R_n(x)$ and if $\lim_{n\to\infty} R_n(x) = 0$, then

$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x)$.

◎ Thm> If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f at a, and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

◎ Taylor's Inequality

If $|f^{(n+1)}(x)| < M$ for $|x-a| \le d$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$.

◎ Note

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for $\forall x \in \mathbb{R} \Rightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0$, $\forall x \in \mathbb{R}$

$f(x) = e^x$, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, let $a_n = \frac{x^n}{n!}$ then

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$, $|f^{(n+1)}(x)| \le e^d$, $|x| \le d$

By Taylor's Inequality, $|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \to 0$

$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$, $|x| \le d$ and $\forall x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

◎ Taylor's Theorem

Suppose that the derivatives $f^{(k)}$ for $k = 0, 1, 2, \cdots, n$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b). Let $x_0 \in [a, b]$. Then for each $x \in [a, b]$,

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(x-x_0)^n + R_n(x)$

and $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ for some c between x and x_0.

<Example> If $a=2$, then $e^x=\sum_{n=0}^{\infty}\frac{e^2}{n!}(x-2)^n$.

<Example> $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

$\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots = \sum_{n=1}^{\infty}\frac{U_{2n+1}x^{2n+1}}{(2n+1)!}$

$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!}$ $(|x|<\frac{\pi}{2})$

<Example> If $a=\frac{\pi}{3}$, then $\sin x = \sum_{n=0}^{\infty}(\sqrt{3})^{a_n}(-1)^{b_n}\frac{1}{2n!}\left(x-\frac{\pi}{3}\right)^n$

$(a_n=\frac{1+(-1)^n}{2},\ b_n=\frac{1}{2}-\frac{i^n}{2i^{\frac{1-(-1)^n}{2}}})$

◎ $f(x)=(1+x)^k,\ k\in\mathbb{R}$

$f'(x)=k(1+x)^{k-1},\ f''(x)=k(k-1)(1+x)^{k-2},\ f^{(n)}(x)=k(k-1)\cdots(k-n+1)x^{k-n}$
$f'(0)=k,\ f''(0)=k(k-1),\ \cdots,\ f^{(n)}(0)=k(k-1)\cdots(k-n+1)$

Therefore, the Mclaurin series of $f(x)=(1+x)^k$ is
$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{k(k-1)\cdots(k-n+1)}{n!}x^n=\sum_{n=0}^{\infty}\binom{k}{n}x^n$, and

$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n$ for $|x|<1$ (though it is difficult to prove).

$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots$

<Example> $f(x)=\frac{1}{\sqrt{4-x}}=\frac{1}{2}\frac{1}{\sqrt{1-\frac{x}{4}}}=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-\frac{x}{4}\right)^n$

$=\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n 1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2n-1)x^n}{n!2^n4^n}$

$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{{}_{2n}\mathrm{C}_n}{2^{4n+1}}x_n$ $\left|-\frac{x}{4}\right|<1,\ |x|<4.$

<Example> $f(x) = x\cos x = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!},\ R = \infty.$

<Example> $f(x) = \ln(1+3x^2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(3x^2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n x^{2n}}{n},\ |x| < \frac{1}{\sqrt{3}}.$

<Example> $\int_0^1 e^{-x^2}dx = \int_0^1\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}dx = \int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}dx$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!} = \frac{1}{2} - \frac{1}{1!\times 3} + \frac{1}{2!\times 5} - \frac{1}{3!\times 7} + \cdots$$

<Example> $\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{1}{x^2}\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right) = \frac{1}{2}$

<Example> $e^x\sin x = \left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]$

<Example> $\tan x = \frac{\sin x}{\cos x} = \left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\right]^{-1}$