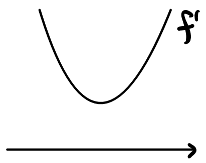
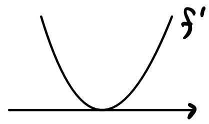
최고차항의 계수가 $\frac{1}{2}$인 삼차함수 $f(x)$와 실수 $t$에 대하여 방정식 $f'(x)=0$이 닫힌구간 $[t, t+2]$에서 갖는 실근의 개수를 $g(t)$라 할 때, 함수 $g(t)$는 다음 조건을 만족시킨다.

> (가) 모든 실수 $a$에 대하여 $\lim_{t\to a+} g(t) + \lim_{t\to a-} g(t) \le 2$이다.
>
> (나) $g(f(1)) = g(f(4)) = 2$, $g(f(0)) = 1$

$f(5)$의 값을 구하시오. [4점] 9

sol.)

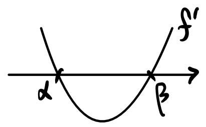



∴ $g(\ )=2$ 일 수 없음.

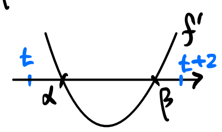
∴ $\lim_{t\to a+} g(t) + \lim_{t\to a-} g(t) = 4$ 일 때 있음.

∴



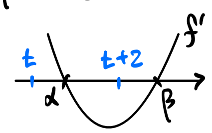

위와 같은 꼴이어야 함.

i) $\beta-\alpha < 2$



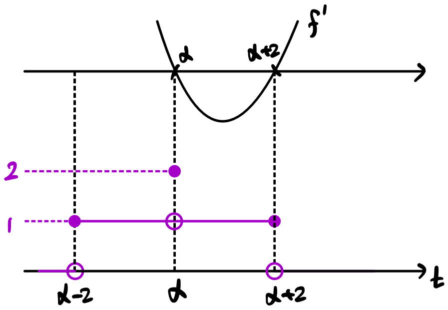

$\lim_{t\to a+} g(t) + \lim_{t\to a-} g(t) = 4$ 일 때 있음.

ii) $\beta-\alpha > 2$




$g(\ ) = 0$ 또는 1

$\therefore \beta - \alpha = 2$




$\longrightarrow$ 가) 성립

$f(1) = f(4) = \alpha$

$\alpha - 2 \le f(0) < \alpha$ 또는 $\alpha \le f(0) < \alpha + 2$

$\therefore f'(x) = \frac{3}{2}(x-\alpha)(x-\alpha-2)$

$f(x) = \frac{1}{2}x^3 - \frac{3}{2}(\alpha+1)x^2 + \frac{3}{2}(\alpha^2 + 2\alpha)x + C$

$f(1) = f(4)$

$\longrightarrow$ $\alpha = 1$ 또는 $\alpha = 2$

i) $\alpha = 2$

$f(x) = \frac{1}{2} - \frac{9}{2}x^2 + 12x + C$

$f(1) = C + 8 = 2 \quad (\because f(1) = f(4) = \alpha)$

$\therefore C = -6$

$g(f(0)) = g(-6) = 0$

$\longrightarrow$ 모순

ii) $\alpha=1$

$f(x)=\frac{1}{2}x^3-3x^2+\frac{9}{2}x+C$

$f(1)=C+2=1$

$\therefore C=-1$

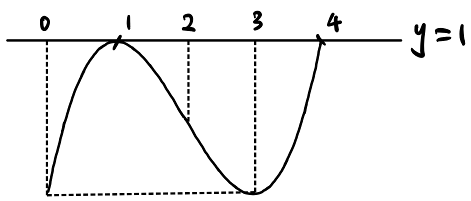
$g(f(0))=g(-1)=1$

$\therefore f(x)=\frac{1}{2}x^3-3x^2+\frac{9}{2}x-1$

$\therefore f(5)=9$

sol2) 현장풀이 — 직관...

$g(f(1))=g(f(4))$ : $f(1)=f(4)$ $\longrightarrow$ 극값...?




$\longrightarrow f(x)=\frac{1}{2}(x-1)^2(x-4)+1$

$\therefore f(5)=9$