38

2019학년도 수능(가형) 30번

최고차항의 계수가 6π인 삼차함수 $f(x)$에 대하여 함수 $g(x)=\dfrac{1}{\underbrace{2+\sin(f(x))}_{>0}}$이 $x=\alpha$에서 극대 또는 극소이고, $\alpha \geq 0$인 모든 α를 작은 수부터 크기순으로 나열한 것을 α_1, α_2, α_3, α_4, α_5, $\cdots$라 할 때, $g(x)$는 다음 조건을 만족시킨다.

(가) $\alpha_1=0$이고 $g(\alpha_1)=\dfrac{2}{5}$이다.
(나) $\dfrac{1}{g(\alpha_5)}=\dfrac{1}{g(\alpha_2)}+\dfrac{1}{2}$

$g'\left(-\dfrac{1}{2}\right)=a\pi$라 할 때, a^2의 값을 구하시오. (단, $0<f(0)<\dfrac{\pi}{2}$) [4점]27

sol.)

$$g'(x)=-\frac{\cos(f(x))\cdot f'(x)}{\{2+\sin(f(x))\}^2}$$

$g(0)=\dfrac{2}{5}$, $g'(0)=0$ $(\because \alpha_1=0)$

$\therefore \dfrac{1}{2+\sin(f(0))}=\dfrac{2}{5} \longrightarrow \sin(f(0))=\dfrac{1}{2}$

$\longrightarrow f(0)=\dfrac{\pi}{6}$ $\left(0<f(0)<\dfrac{\pi}{2}\right)$

$\therefore -\dfrac{\cos(f(0))\cdot f'(0)}{\{2+\sin(f(0))\}^2}=0 \longrightarrow \dfrac{\sqrt{3}}{2}f'(0)=0$

$\longrightarrow f'(0)=0$

$2+\sin(f(\alpha_5))=2+\sin(f(\alpha_2))+\dfrac{1}{2}$

$\therefore \sin(f(\alpha_5))=\sin(f(\alpha_2))+\dfrac{1}{2}$

If) $g'(\alpha_5)=0$ 에서 $\cos(f(\alpha_5))=0$ 이라면

$-\frac{1}{2} \le \sin(f(\alpha_5)) \le \frac{3}{2} \quad \left(\because -1 \le \sin(f(\alpha_2)) = \sin(f(\alpha_5)) - \frac{1}{2} \le 1\right)$

$\therefore \sin(f(\alpha_5))=1, \quad \sin(f(\alpha_2))=\frac{1}{2}$

$\longrightarrow \cos(f(\alpha_2)) \neq 0$

$\longrightarrow f'(\alpha_2)=0$

$f(\alpha_2) < \frac{\pi}{6} \quad (\because f(\alpha_2) : 극소)$

$\therefore \max(f(\alpha_2)) = -\frac{7}{6}\pi$

$(0, \alpha_2)$ 사이에 $f(x) = -\frac{\pi}{2}$를 만족시키는 x값 존재

$\longrightarrow$ 모순

$\therefore \cos(f(\alpha_2))=0, \quad \sin(f(\alpha_2))=-1, \quad \sin(f(\alpha_5))=-\frac{1}{2}$

$\therefore f(\alpha_2)=-\frac{\pi}{2}, \quad f'(\alpha_5)=0$

$\therefore f(\alpha_3)=-\frac{3}{2}\pi, \quad f(\alpha_4)=-\frac{5}{2}\pi, \quad f(\alpha_5)=-\frac{17}{6}\pi \quad \left(\because -\frac{7}{2}\pi \le f(\alpha_5) < -\frac{5}{2}\pi\right)$

$f'(x) = 18\pi x(x-\alpha_5)$

$\therefore f(x) = 6\pi x^3 - 9\alpha_5 \pi x^2 + C$

$f(0) = \frac{\pi}{6} = C \longrightarrow f(x) = 6\pi x^3 - 9\alpha_5 \pi x^2 + \frac{\pi}{6}$

$f(\alpha_5) = 6\pi(\alpha_5)^3 - 9\pi(\alpha_5)^3 + \frac{\pi}{6} = -\frac{17}{6}\pi$

$\therefore \alpha_5 = 1, \quad f(x) = 6\pi x^3 - 9\pi x^2 + \frac{\pi}{6}, \quad f'(x) = 18\pi x^2 - 18\pi x$

$$g'\left(-\frac{1}{2}\right) = \frac{\cos\left(-\frac{17}{6}\pi\right)\cdot\frac{27}{2}\pi}{\left\{2+\sin\left(-\frac{17}{6}\pi\right)\right\}^2} = 3\sqrt{3}\pi$$

$\therefore a = 3\sqrt{3}, \quad a^2 = 27$

Sol$_2$) 삼차함수 비율관계

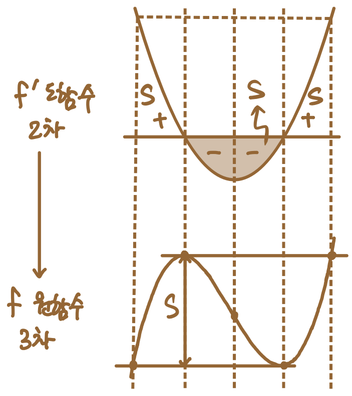
sol$_1$) 에서 $f(\alpha_5)$가 $-\frac{17}{6}\pi$인 것을 알아내고

바로 $f(x)$를 만들 수 있다.

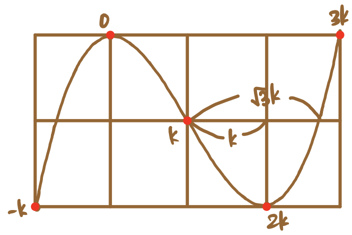
$$f(x) - \frac{1}{6}\pi = 6\pi x^2\left(x - \frac{3}{2}\alpha_5\right)\Big|_{x=\alpha_5} \Rightarrow \alpha_5 = 1$$

⋮

* 참고 : 삼차함수 비율관계



x좌표 등차



합동인 직사각형 8개

38

sol3) 시각적인 해석

$2+\sin(f(x)) \ge 1$

$\longrightarrow \frac{1}{g(x)} = 2+\sin(f(x))$: $x=\alpha$ 에서 여전히 극대·극소를 가짐.

$\longrightarrow \frac{1}{g(x)}$ 를 관찰해도 됨!

$f(x)$의 그래프를 적절히 회전시켜서 정의역처럼 생각해보자.

if) $f(x)$가 극값 X

: 극소: 1, 극대: 3

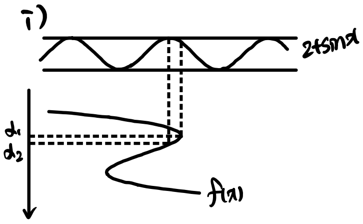
$\longrightarrow$ 나 조건 모순 $\left(\because 3 \neq 3+\frac{1}{2},\ 3 \neq 1+\frac{1}{2} \cdots\right)$

$\longrightarrow f(x)$: 극값을 두개 갖는다! + 3/1 이 다른 극점

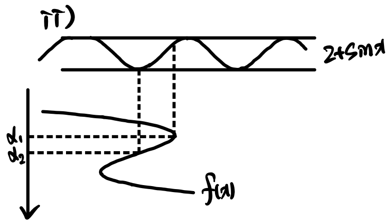
$\longrightarrow f$의 자체극값

$2+\sin(f(x))$ 가 $x=\alpha_1=0$ 일 때 $\frac{5}{2}$ 이다.

i)



ii)

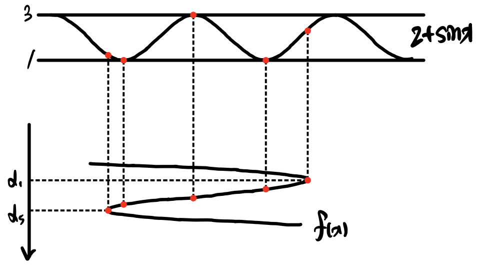


i) $x=\alpha_2$ 에서 극댓값 3

$\frac{1}{g(\alpha_5)} = \frac{1}{g(\alpha_2)} + \frac{1}{2} = 3+\frac{1}{2} > 3$: 모순 $\left(\because 1 \le \frac{1}{g(x)} \le 3\right)$

$\therefore$ ii) $x=\alpha_2$ 에서 극솟값 1

$\frac{1}{g(\alpha_5)} = 1+\frac{1}{2} = \frac{3}{2}$: 자체 극값 $(\because \neq 1, 3)$



$2+\sin(f(\alpha_1))=\frac{5}{2}$

$\therefore f(\alpha_1)=\frac{\pi}{6}$

$2+\sin(f(\alpha_5))=\frac{3}{2}$, $\therefore \sin(f(\alpha_5))=-\frac{1}{2}$

$\therefore f(\alpha_5)=-\frac{17}{6}\pi$ ($\because \sin x=-\frac{1}{2}$ 인 음의 x값 중 4번째로 큰 수)

$\therefore f(x)$: $x=0$, 극대 $\frac{\pi}{6}$

$x=\alpha_5$, 극소 $-\frac{17}{6}\pi$

$f(x)-\frac{1}{6}\pi=6\pi x^2(x-3p)$

$f(2p)=-\frac{17}{6}\pi$ ($\because$ $2:1 \longrightarrow$ 비율관계)

$\therefore p=\frac{1}{2}$, $f(x)=6\pi x^2\left(x-\frac{3}{2}\right)+\frac{1}{6}\pi$

⋮

38

sol4) 단위원 해석

$$\frac{1}{g(x)} - x = \sin(f(x)) = \sin x \circ f(x) = h(x)$$

가: $\alpha_1 = 0$ 이고, $h(\alpha_1) = \frac{1}{2}$ 이다.

나: $h(\alpha_5) - h(\alpha_2) = \frac{1}{2}$

: $f(x)$를 단위원 위의 동경값으로,

$h(x)$를 단위원 위의 y좌표로 해석

$h(x) = \pm 1$: 겉함수 ($\sin x$)가 만든 극값

$h(x) \neq \pm 1$: 속함수 ($f(x)$)가 만든 극값

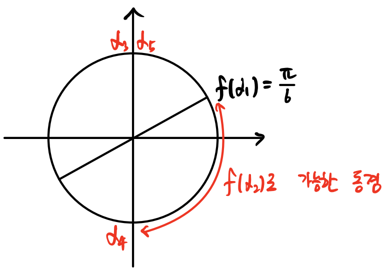
if) $f(x)$: $x = \alpha_1$ 에서 극소

: 극소 $\longrightarrow$ 극대 사이에서 증가

$\longrightarrow$ 나 조건 모순

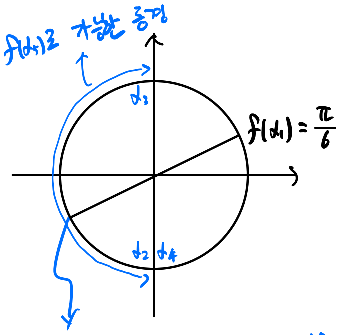
$\longrightarrow$ $f(x)$: $x = \alpha_1$ 에서 극대 + 나) 조건에서 α_5와 α_2 중 하나는 $f(x)$의 극값이다.

if) $f(x)$: $x = \alpha_2$ 에서 극소



$$\underbrace{h(\alpha_5)}_{=1} - h(\alpha_2) = \frac{1}{2} \longrightarrow h(\alpha_2) = \frac{1}{2} \longrightarrow \text{모순}$$

$\longrightarrow$ $f(x)$: $x = \alpha_5$ 에서 극소



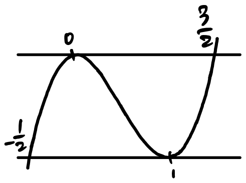
$f(\alpha_5) = -2\pi - \frac{5}{6}\pi = -\frac{17}{6}\pi$

$\therefore\ h(\alpha_5) = \frac{1}{2}$

$f(\alpha_1) - f(\alpha_5) = \int_{\alpha_5}^{\alpha_1} f'(x)\,dx$

$= 3\pi = \frac{18}{6}\pi \cdot (\alpha_5)^3$

$\therefore\ \alpha_5 = 1$



$f(x) = 6\pi\left(x+\frac{1}{2}\right)(x-1)^3 - \frac{17}{6}\pi$

$\vdots$