

부정적분 200題

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(1) 각변환

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\sin(\pi + \theta) = -\sin\theta$$

$$\cos(\pi + \theta) = -\cos\theta$$

$$\tan(\pi + \theta) = \tan\theta$$

$$\sin(\pi - \theta) = \sin\theta$$

$$\cos(\pi - \theta) = -\cos\theta$$

$$\tan(\pi - \theta) = -\tan\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

$$\sin\left(\frac{3}{2}\pi + \theta\right) = -\cos\theta$$

$$\cos\left(\frac{3}{2}\pi + \theta\right) = \sin\theta$$

$$\tan\left(\frac{3}{2}\pi + \theta\right) = -\cot\theta$$

$$\sin\left(\frac{3}{2}\pi - \theta\right) = -\cos\theta$$

$$\cos\left(\frac{3}{2}\pi - \theta\right) = -\sin\theta$$

$$\tan\left(\frac{3}{2}\pi - \theta\right) = \cot\theta$$

(2) 덧셈정리

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \quad (1 \mp \tan\alpha\tan\beta \neq 0)$$

(3) 배각공식

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \quad (1 - \tan^2\alpha \neq 0)$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \quad (1 - 3\tan^2\alpha \neq 0)$$

(4) 반각공식

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$$

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$$

$$\tan^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{1 + \cos\alpha}$$

(5) 곱을 합 또는 차로 고치는 공식

$$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$$

$$\cos\alpha\sin\beta = \frac{1}{2}\{\sin(\alpha+\beta) - \sin(\alpha-\beta)\}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$$

(6) 합 또는 차를 곱으로 고치는 공식

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

(7) 삼각함수의 합성

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cos(\theta-\beta)$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right), \quad \beta = \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2} - \alpha$$

(8) 삼각함수 항등식

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

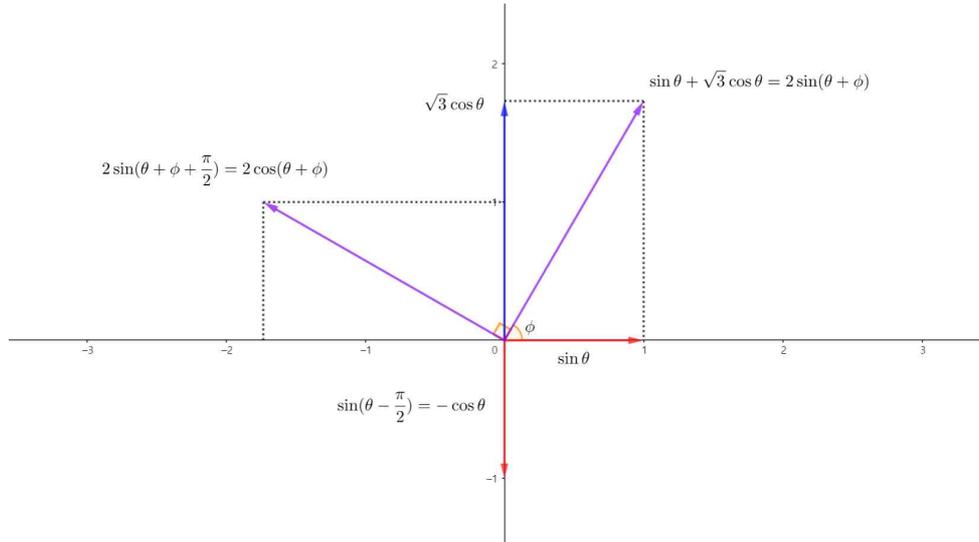
$$\cot^2\theta + 1 = \csc^2\theta$$

(9) 부정적분과 미분

$$\textcircled{1} \frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

$$\textcircled{2} \int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C$$

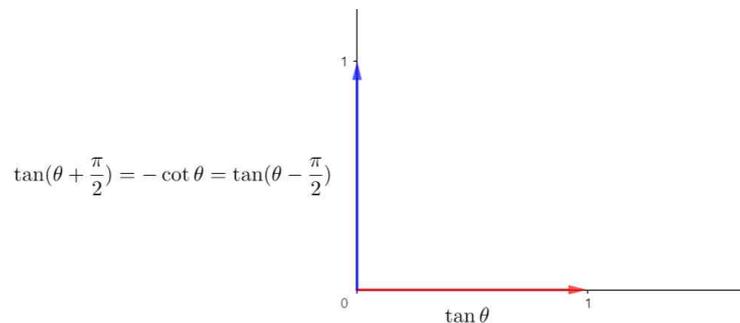
(10) 삼각함수 위상자(Phasor)



- x 축의 양의 방향을 \sin 축, y 축의 양의 방향을 \cos 축으로 설정하여 삼각함수의 위상을 벡터로 표현한다.
- 각이 더해질 경우 위상자는 길이는 유지된 채 반시계방향으로 회전한다.
- 위상자가 표현하는 삼각함수의 계수는 그 위상자의 길이로 표현된다.
- 서로 다른 두 위상자를 벡터합하면 이는 각 위상자가 표현하는 삼각함수의 합성과 같다.

가령, 위 사진에서 $\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \phi)$ 이고 $\phi = \frac{\pi}{3}$ 이다.

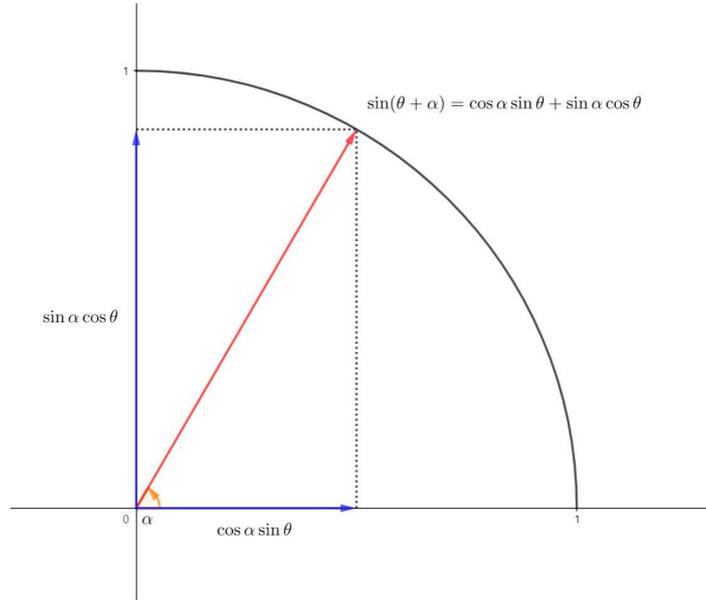
- $\sin\theta$ 위상자를 시계방향으로 $\frac{\pi}{2}$ 만큼 회전시키면 $\sin(\theta - \frac{\pi}{2})$ 이며, 이는 $-\cos$ 축이므로 $\sin(\theta - \frac{\pi}{2}) = -\cos\theta$ 이다.
- \sin 을 \csc 로, \cos 을 \sec 로 바꾸면 \csc 와 \sec 에 대한 각변환이 가능하나 덧셈정리와 합성은 성립하지 않는다. 즉, $\csc\theta + \sqrt{3}\sec\theta = 2\csc(\theta + \phi)$ 는 성립하지 않는다.



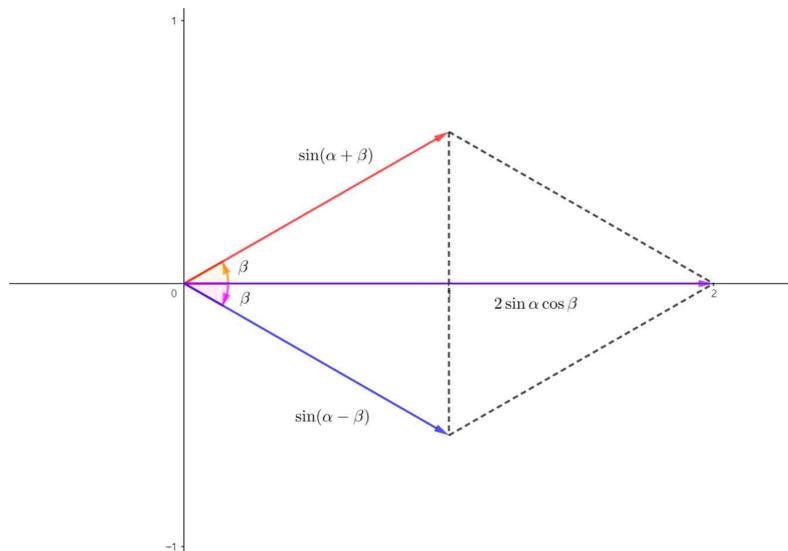
- \tan 와 $-\cot$ 의 경우 위와 같이 xy 평면의 제 1사분면만을 이용하여 도시할 수 있다. 이 경우 주기가 π 이므로 시계방향으로 회전하는 경우 돌아간 각도를 $\frac{3}{2}\pi$ 가 아닌 $\frac{\pi}{2}$ 로 본다.

(11) 삼각함수 위상자의 활용

- 삼각함수 위상자를 사용하면 (1), (2), (5), (6), (7)의 공식들은 모두 증명 가능하다. 이에 대한 예시로 (2)와 (5)의 첫 번째 공식의 증명을 제시해 놓는다.



그림과 같은 단위원에서 $\sin(\theta + \alpha)$ 는 $\sin\theta$ 축에서 길이 1인 위상자가 각도 α 만큼 회전한 것이다. 따라서 위상자의 종점에서 \sin 축, \cos 축에 각각 수선의 발을 내리면 원점을 시점으로 하고 두 수선의 발을 종점으로 하는 두 위상자의 길이는 각각 $\cos\alpha$, $\sin\alpha$ 이다. 즉 처음의 $\sin(\theta + \alpha)$ 위상자(빨간색)를 \sin 축 성분과 \cos 축 성분의 두 위상자(파란색)로 분해할 수 있고 이들의 길이는 각각 $\cos\alpha$, $\sin\alpha$ 이므로, $\sin(\theta + \alpha) = \cos\alpha\sin\theta + \sin\alpha\cos\theta$ 가 성립한다.



그림과 같이 기준각이 α 인 위상 평면에서 $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$ 는 길이가 1인 $\sin\alpha$ 축 위상자가 각각 β , $-\beta$ 만큼 회전한 것이다. 따라서 이들을 합성한 보라색 위상자는 $\sin\alpha$ 축으로 길이가 $2\cos\beta$ 인 위상자이므로 $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$ 이고 증명이 완료되었다.

이와 비슷한 방법으로 (1), (2), (5), (6), (7)의 공식들을 모두 증명할 수 있다.

(12) 부정적분의 기본 성질

$$\textcircled{1} \int kf(x)dx = k \int f(x)dx \quad (k \in \mathbb{R})$$

$$\textcircled{2} \int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx \quad (\text{복부호동순})$$

(13) 치환적분법

$$\textcircled{1} g(x) = t \text{ 일 때, } \int f(g(x))g'(x)dx = \int f(t)dt$$

$$\textcircled{2} \int f(x)dx = F(x) + C \text{ 일 때, } \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

$$\textcircled{3} \int \frac{1}{ax+b}dx = \frac{1}{a} \ln|ax+b| + C$$

$$\textcircled{4} \int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$$

(14) 바이어슈트라스 치환 (Weierstrass Substitution)

- 바이어슈트라스 치환은 삼각함수의 유리 적분을 유리식의 적분으로 바꿔주는 치환법이다.

$$\tan \frac{x}{2} = t$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}$$

(15) 오일러 치환 (Euler's Substitution)

- 오일러 치환은 유리 이변수 함수 R 에 대하여 다음과 같은 부정적분을 계산하기 위한 치환법이다.

$$\int R(x, \sqrt{ax^2 + bx + c})dx$$

[1] 제 1종 오일러 치환

$a > 0$ 일 때,

$$\sqrt{ax^2 + bx + c} = \pm x \sqrt{a} + t,$$

$$x = \frac{c-t^2}{\pm 2t \sqrt{a-b}}$$

와 같이 치환한다.

[2] 제 2종 오일러 치환

$c > 0$ 일 때,

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c},$$

$$x = \frac{\pm 2t\sqrt{c} - b}{a - t^2}$$

와 같이 치환한다.

[3] 제 3종 오일러 치환

방정식 $ax^2 + bx + c = 0$ 이 두 실근 α, β 를 가질 때,

$$\sqrt{ax^2 + bx + c} = \sqrt{a(x - \alpha)(x - \beta)} = (x - \alpha)t,$$

$$x = \frac{\alpha\beta - \alpha t^2}{a - t^2}$$

와 같이 치환한다.

(16) 부정적분 공식

1. $\int dx = x + C$

2. $\int adx = ax + C \quad (a \in \mathbb{R})$

3. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (-1 \neq n \in \mathbb{R})$

4. $\int \frac{1}{x} dx = \ln|x| + C$

5. $\int e^x dx = e^x + C$

6. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (0 < a \neq 1)$

7. $\int \ln x dx = x \ln x - x + C$

8. $\int \sin x dx = -\cos x + C$

9. $\int \cos x dx = \sin x + C$

10. $\int \tan x dx = \ln|\sec x| + C$

11. $\int \csc x dx = \ln|\csc x - \cot x| + C$

12. $\int \sec x dx = \ln|\sec x + \tan x| + C$

13. $\int \cot x dx = \ln |\sin x| + C$
 14. $\int \sec^2 x dx = \tan x + C$
 15. $\int \csc^2 x dx = -\cot x + C$
 16. $\int \sec x \tan x dx = \sec x + C$
 17. $\int \csc x \cot x dx = -\csc x + C$
 18. $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$
 19. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + C$
 20. $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln |x + \sqrt{x^2 \pm a^2}| + C$
 21. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
 22. $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$
 23. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$
 24. $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(17) 헤비사이드 법 (부분분수 분해)

$$\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^n \frac{b_i}{x-a_i} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n} \text{ 일 때}$$

$$\frac{f(x)}{(x-a_i)} = h_i(x) \text{ 라 하면 } b_i = \frac{g(a_i)}{h_i(a_i)} \text{ 가 성립한다. } (1 \leq i \leq n)$$

(18) 부분적분법

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

(19) 삼각함수의 거듭제곱의 부정적분 공식 (Reduction Formula)

$n \in \mathbb{N} - \{1\}$,

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

※ 부정적분은 원시함수들의 집합이므로 예시 답과 다르다고 틀린 것이 아닙니다. Wolfram Alpha 등을 이용하여 미분했을 때 맞게 나온다면 정답입니다. 다만 모범 답안에는 정석적인 방법론으로 최대한 간단하게 풀이하였으니 모범 답안과 다르게 나온 경우에는 답안의 풀이도 한 번 읽어보시길 바랍니다.

※※ 쌍곡선함수($\sinh x$, $\cosh x$, $\tanh x$)와 그 역함수는 등장하지 않으며, 역쌍곡선함수의 경우 $\tanh^{-1}x$ 라는 표기 대신 $\frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$ 등으로 나타내었습니다. 적분 결과 또는 피적분함수에 역삼각함수가 등장하는 경우 고등학교 교육과정을 벗어나는 것이기는 하나, (16)의 19, 21번 공식을 사용하는 수준에서 모두 해결 가능합니다.

역삼각함수가 등장하는 문항은 다음과 같습니다. (문제 번호를 빨간색으로 표시했습니다.)

25, 29, 43, 65, 68, 69, 79, 83, 85, 86, 90, 103, 104, 105, 106, 108, 109, 111, 143, 144, 146, 151, 153, 159
161, 164, 165, 166, 168, 169, 173, 174, 175, 176, 178, 179, 180, 182, 183, 184, 185, 191, 193, 194, 196, 197, 199

[1~200] 다음 부정적분을 구하시오.

1. $\int \frac{x^3 - 2x + 1}{\sqrt[3]{x}} dx$
2. $\int \cos^2 \frac{x}{2} dx$
3. $\int 2^{x-1} dx$
4. $\int \tan x dx$
5. $\int \sin x \cos^2 x dx$
6. $\int \sec^2 x dx$
7. $\int \frac{3}{3x+2} dx$
8. $\int x \sqrt{x^2+1} dx$
9. $\int \frac{x^2}{x-2} dx + \int \frac{3x-10}{x-2} dx$
10. $\int \csc x (\sin x + \cot x) dx$
11. $\int 2^{2x} dx$
12. $\int \frac{e^{2x} - 4}{e^x - 2} dx$
13. $\int \sin 3x \cos 4x dx$
14. $\int \sin x \cos 3x dx$

15. $\int \cos 2x \cos 6x dx$
16. $\int \sin^4 x dx$
17. $\int \cos^4 x dx$
18. $\int \sin^6 x \cos^3 x dx$
19. $\int \sin^3 x \sec^2 x dx$
20. $\int \sin^4 x \cos^4 x dx$
21. $\int \sin^2 x \cos^5 x dx$
22. $\int \sin^3 x \cos^5 x dx$
23. $\int \sin^2 x \cos^4 x dx$
24. $\int \sin^4 x \cos^2 x dx$
25. $\int \sqrt{1-4x^2} dx$
26. $\int \frac{1}{x^2 \sqrt{x^2+4}} dx$
27. $\int \frac{1}{(x+1) \sqrt{x^2+2x+2}} dx$
28. $\int \frac{x^2+2x-2}{x^3-4x} dx$
29. $\int \frac{1}{x^4-1} dx$
30. $\int (x^2+2x+2)^5 (x+1) dx$
31. $\int (x^2-1) \sqrt{x^3-3x} dx$
32. $\int (1+\sin x)^2 \cos x dx$
33. $\int \sin x \cos 2x dx$
34. $\int \frac{e^x}{\sqrt{e^x+1}} dx$
35. $\int \frac{2x+1}{x^2+x+1} dx$
36. $\int \frac{x+1}{x^3-1} dx$
37. $\int \tan 2x dx$
38. $\int \frac{1}{x \ln x} dx$

39. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$
40. $\int (2 + 3x) \sqrt{1 + 2x} dx$
41. $\int \frac{x-1}{\sqrt{x+1}} dx$
42. $\int \frac{x^3}{\sqrt{1-x^2}} dx$
43. $\int \frac{1}{1 + \cos^2 x} dx$
44. $\int \frac{x}{(2x+1)^3} dx$
45. $\int \frac{1}{x^3(x+1)} dx$
46. $\int \frac{1}{1 + \cos x} dx$
47. $\int \frac{x}{\sqrt{1-x^2}} dx$
48. $\int \frac{e^x - 1}{e^x + 1} dx$
49. $\int \frac{1}{e^{2x} + 1} dx$
50. $\int \frac{1}{e^{2x} + e^x} dx$
51. $\int \frac{e^x \ln(e^x + 1)}{e^x + 1} dx$
52. $\int \frac{e^x (e^x + 1)}{(e^x + 3)^2} dx$
53. $\int \frac{\ln x}{x(\ln x + 1)^2} dx$
54. $\int \frac{x^4}{(x-1)^3} dx$
55. $\int \frac{x^2}{(x^2 - a^2)(x^2 - b^2)} dx \quad (a \neq b)$
56. $\int x^3 \sqrt{x^2 + a} dx$
57. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3 + 1}} dx$
58. $\int \frac{1}{1 + \tan x} dx$
59. $\int \frac{1}{\sin 2x - \sin x} dx$

60. $\int \frac{1}{\sin x} dx$
61. $\int \frac{1}{1 - 3\cos^2 x} dx$
62. $\int \frac{\sin x}{1 + \sin x} dx$
63. $\int \sqrt{1 + \sqrt{x}} dx$
64. $\int \sqrt{x + \sqrt{x^2 + 3}} dx$
65. $\int \frac{x}{\sqrt{5 - 4x - x^2}} dx$
66. $\int \frac{x^3 - x - 2}{x^3 - x^2 + x - 1} dx$
67. $\int \frac{1}{x^2 + 1} dx$
68. $\int \frac{1}{ax^2 + bx + c} dx \quad (b^2 - 4ac < 0, a > 0)$
69. $\int \frac{1}{x^2 + x + 1} dx$
70. $\int \frac{1}{3 + 5\cos x} dx$
71. $\int \frac{1}{4\sin x + 3\cos x} dx$
72. $\int \frac{1}{\sqrt{(a^2 - x^2)^3}} dx \quad (a > 0)$
73. $\int \frac{1}{\sqrt{(a^2 + x^2)^3}} dx \quad (a > 0)$
74. $\int \sin^2 x dx$
75. $\int \sin^3 x dx$
76. $\int \tan^2 x dx$
77. $\int \tan^3 x dx$
78. $\int \sec^3 x dx$
79. $\int \frac{1}{4x^2 + 4x + 2} dx$
80. $\int \frac{1}{\sqrt{x^2 + 1}} dx$
81. $\int \frac{1}{\sqrt{x^2 - 4}} dx$

82. $\int \frac{1}{\sqrt{x^2 - 2x + 2}} dx$
83. $\int \frac{\sqrt{9 - x^2}}{x^2} dx$
84. $\int \frac{4x}{(x-1)^2(x+1)} dx$
85. $\int \frac{2x^2 - x + 4}{x(x^2 + 4)} dx$
86. $\int \frac{1}{16x^4 - 1} dx$
87. $\int \frac{x^2 + 1}{(x-1)(x-2)(x-3)} dx$
88. $\int \frac{x + 4}{x^3 + 3x^2 - 10x} dx$
89. $\int \sqrt{x^2 + 1} dx$
90. $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$
91. $\int \frac{1}{x(1 + x^2)} dx$
92. $\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx$
93. $\int \frac{1}{x \sqrt{1 - x^2}} dx$
94. $\int \frac{1}{\sqrt{(x - \alpha)(x - \beta)}} dx \quad (\alpha, \beta \in \mathbb{R})$
95. $\int \frac{1}{3\cos^2 x - \sin^2 x} dx$
96. $\int \frac{7x^3 - 13x^2 - 24x + 24}{x^4 - 3x^3 - 10x^2 + 24x} dx$
97. $\int \frac{1}{\sqrt{x} + \sqrt[4]{x^3}} dx$
98. $\int \frac{1}{x^2 \sqrt{2x - x^2}} dx$
99. $\int \frac{1}{x + \sqrt{x^2 - 1}} dx$
100. $\int \frac{1}{(x-1)^2 \sqrt{3 + 2x - x^2}} dx$
101. $\int \frac{x^4 - 1}{x^2 \sqrt{x^4 - x^2 + 1}} dx$
102. $\int \frac{x^4 + 81}{x(x^2 + 9)^2} dx$

103. $\int \frac{x^2 - x + 2}{x^3 - 1} dx$
104. $\int \sqrt{\frac{4-x}{x}} dx$
105. $\int \sqrt{\frac{x}{1-x^3}} dx$
106. $\int \sqrt{x} \sqrt{1-x} dx$
107. $\int \sqrt{\frac{x-2}{x-1}} dx$
108. $\int \frac{\sqrt{x^2-25}}{x^3} dx$
109. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$
110. $\int \frac{1}{\sin^2 x \cos^2 x} dx$
111. $\int \frac{1}{(1+x^2)^2} dx$
112. $\int \tan^6 x \sec^4 x dx$
113. $\int \tan^4 x \sec^4 x dx$
114. $\int \tan x \sec^5 x dx$
115. $\int \tan^5 x \sec^7 x dx$
116. $\int x \sec^2 x dx$
117. $\int x \sin(x+2) dx$
118. $\int \ln(x^2-x) dx$
119. $\int e^x \sin^2 \frac{x}{2} dx$
120. $\int \frac{2x+7}{\sqrt{x^2+6x-7}} dx$
121. $\int \frac{2x-1}{x^2-2x-3} dx$
122. $\int 4 \sin 2x \cos 2x \cos 4x dx$
123. $\int (2x+3)e^x dx$
124. $\int x \ln x dx$
125. $\int x^2 e^x dx$

126. $\int x^2 \ln x dx$
127. $\int x \cos x dx$
128. $\int (\ln x)^2 dx$
129. $\int e^x \sin x dx$
130. $\int e^x \cos x dx$
131. $\int e^x \sin^2 x dx$
132. $\int x^n \ln x dx \quad (n \in \mathbb{Z})$
133. $\int \ln x dx$
134. $\int \ln(x+1) dx$
135. $\int \frac{1}{1+e^x} dx$
136. $\int \frac{x^3 + x^2 + 1}{x^2(x^2 + 1)} dx$
137. $\int (x^2 + 1)e^x dx$
138. $\int \frac{1}{x\sqrt{x+1}} dx$
139. $\int \frac{x\sqrt{x}-1}{x-\sqrt{x}} dx$
140. $\int \ln(x + \sqrt{x^2 + 1}) dx$
141. $\int \left(x^2 - \frac{1}{x^2}\right) \left(x - \frac{1}{x}\right) dx$
142. $\int \sin \sqrt{x} dx$
143. $\int \ln(x^2 + 1) dx$
144. $\int \frac{-2x}{\sqrt{1-x^4}} dx$
145. $\int e^{\sqrt{x}} dx$
146. $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx$
147. $\int \frac{1}{x\sqrt{a^2 - x^2}} dx$
148. $\int \frac{1}{\sqrt{1+e^x}} dx$

149. $\int \frac{x}{\sqrt{1+x^4}} dx$
150. $\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx$
151. $\int \frac{x+2}{(x^2+1)(x-1)^3} dx$
152. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}} dx$
153. $\int \frac{\tan^{-1}x}{1+x^2} dx$
154. $\int \frac{4}{x^2\sqrt{4-x^2}} dx$
155. $\int 3\sec^4 3x dx$
156. $\int \cot^3 x dx$
157. $\int \csc^4 x dx$
158. $\int \tan^2 3x dx$
159. $\int \frac{3x^4 - 3x^3 - x^2 - 17x - 2}{(x-3)(x^2+1)^2} dx$
160. $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$
161. $\int \frac{1}{x - \sqrt{1-x^2}} dx$
162. $\int \frac{x^2 + 3}{x^2 - 1} dx$
163. $\int 13e^{2x} \cos 3x dx$
164. $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 + 4x} dx$
165. $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$
166. $\int \frac{1}{(x+1)(x^2+1)} dx$
167. $\int \frac{x^2 - 2x - 2}{x^3 - 1} dx$
168. $\int \sqrt{a^2 - x^2} dx \quad (a > 0)$
169. $\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$
170. $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$

171. $\int x \sec^2 x \tan x dx$
172. $\int e^{-x} \sin^2 2x dx$
173. $\int \sqrt{\tan x} dx$
174. $\int \sqrt[3]{\tan x} dx$
175. $\int \left(\frac{x+3}{\sqrt{4-x^2}} + \cot x [\ln(\sin x)] \right) dx$
176. $\int \left(\frac{4x^2}{x^2+9} + \tan^2 2x \right) dx$
177. $\int \frac{1}{(1+x)(2+x)^2(3+x)^3} dx$
178. $\int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$
179. $\int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx$
180. $\int \frac{1}{\cos 2x + 3} dx$
181. $\int e^x \tan x (1 - 2 \sec^2 x) dx$
182. $\int \sqrt{\tan x + 1} dx$
183. $\int \sqrt{\tan x + 2} dx$
184. $\int \frac{\tan^{-1} x}{x \sqrt{x^2 + 1}} \cdot \exp\left(-\frac{\tan^{-1} x}{x}\right) dx$ (단, $\exp x = e^x$)
185. $\int \cos^2(\tan^{-1}(\sin(\cot^{-1} x))) dx$
186. $\int \frac{\sec x - \tan x}{\sqrt{\sin^2 x - \sin x}} dx$
187. $\int \frac{\sec^2 x}{(\sec x + \tan x)^{5/2}} dx$
188. $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$
189. $\int \sin 2022x \cdot \sin^{2020} x dx$
190. $\int \frac{\sqrt{\cos 2x}}{\sin x} dx$
191. $\int \frac{1}{x \sqrt{x^2 + 4x - 4}} dx$
192. $\int \frac{1}{x \sqrt{-x^2 + x + 2}} dx$

193. $\int \frac{x^2}{\sqrt{-x^2+3x-2}} dx$
194. $\int \frac{\sin^3(\theta/2)}{\cos(\theta/2) \cdot \sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}} d\theta$
195. $\int \frac{\tan^4\theta}{1-\tan^2\theta} d\theta$
196. $\int \frac{x^2+1}{x^4+3x^3+3x^2-3x+1} dx$
197. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$
198. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx$
199. $\int \left(\frac{\tan^{-1}x}{1+(x+1/x)\tan^{-1}x} \right)^2 dx$
200. $\int \frac{1}{\prod_{i=0}^m (x+i)} dx = \int \frac{1}{x(x+1)(x+2)\cdots(x+m)} dx \quad (m \in \mathbb{N} \cup \{0\})$

[해설]

1. $t = \sqrt[3]{x}, \quad dx = 3t^2 dt$

$$\int \frac{x^3-2x+1}{\sqrt[3]{x}} dx = \int 3t(t^9-2t^3+1)dt = \frac{3}{11}t^{11} - \frac{6}{5}t^5 + \frac{3}{2}t^2 + C = \frac{3}{11}x^{\frac{11}{3}} - \frac{6}{5}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{2}{3}} + C$$

2. $\int \cos^2 \frac{x}{2} dx = \int \frac{1+\cos x}{2} = \frac{1}{2}x + \frac{1}{2}\sin x + C$

3. $\int 2^{x-1} dx = \frac{1}{\ln 2} 2^{x-1} + C$

4. $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \ln |\sec x| + C$

5. $\int \sin x \cos^2 x dx = -\frac{1}{3} \cos^3 x + C$

6. $\int \sec^2 x dx = \tan x + C$

7. $\int \frac{3}{3x+2} dx = \ln |3x+2| + C$

$$8. t = \sqrt{x^2 + 1}, \quad t dt = x dx$$

$$\int x \sqrt{x^2 + 1} dx = \int t^2 dx = \frac{1}{3} t^3 + C = \frac{1}{3} (x^2 + 1)^{\frac{3}{2}} + C$$

$$9. \int \frac{x^2}{x-2} dx + \int \frac{3x-10}{x-2} dx = \int (x+5) dx = \frac{1}{2} x^2 + 5x + C$$

$$10. \int \csc x (\sin x + \cot x) dx = \int (1 + \csc x \cot x) dx = x - \csc x + C$$

$$11. \int 2^{2x} dx = \frac{1}{2 \ln 2} 2^{2x} + C$$

$$12. \int \frac{e^{2x} - 4}{e^x - 2} dx = \int (e^x + 2) dx = e^x + 2x + C$$

$$13. \int \sin 3x \cos 4x dx = \frac{1}{2} \int (\sin 7x - \sin x) dx = \frac{1}{2} \cos x - \frac{1}{14} \cos 7x + C$$

$$14. \int \sin x \cos 3x dx = \frac{1}{2} \int (\sin 4x - \sin 2x) dx = \frac{1}{4} \cos 2x - \frac{1}{8} \cos 4x + C$$

$$15. \int \cos 2x \cos 6x dx = \frac{1}{2} \int (\cos 8x + \cos 4x) dx = \frac{1}{8} \sin 4x + \frac{1}{16} \sin 8x + C$$

$$16. \int \sin^4 x dx = \frac{1}{4} \int (1 - \cos 2x)^2 dx = \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

$$17. \int \cos^4 x dx = \frac{1}{4} \int (1 + \cos 2x)^2 dx = \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx$$

$$= \int \left(\frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

$$18. \int \sin^6 x \cos^3 x dx = \int \sin^6 x (1 - \sin^2 x) \cos x dx = \int \sin^6 x \cos x dx - \int \sin^8 x \cos x dx$$

$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$$

$$19. \int \sin^3 x \sec^2 x dx = \int \frac{\sin x (1 - \cos^2 x)}{\cos^2 x} dx = \int (\tan x \sec x - \sin x) dx = \sec x + \cos x + C$$

$$20. \int \sin^4 x \cos^4 x dx = \frac{1}{16} \int \sin^4 2x dx = \frac{1}{64} \int (1 - \cos 4x)^2 dx$$

$$= \frac{1}{64} \int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2} \cos 8x \right) dx = \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C$$

$$21. \int \sin^2 x \cos^5 x dx = \int \sin^2 x (1 - \sin^2 x)^2 \cos x dx$$

$$= \int \sin^2 x \cos x dx - 2 \int \sin^4 x \cos x dx + \int \sin^6 x \cos x dx = \frac{1}{7} \sin^7 x - \frac{2}{5} \sin^5 x + \frac{1}{3} \sin^3 x + C$$

$$22. \int \sin^3 x \cos^5 x dx = \int \sin^3 x (1 - \sin^2 x)^2 \cos x dx$$

$$= \int \sin^3 x \cos x dx - 2 \int \sin^5 x \cos x dx + \int \sin^7 x \cos x dx = \frac{1}{8} \sin^8 x - \frac{1}{3} \sin^6 x + \frac{1}{4} \sin^4 x + C$$

$$23. \int \sin^2 x \cos^4 x dx = \frac{1}{4} \int \sin^2 2x \cos^2 x dx = \frac{1}{16} \int (\sin 3x + \sin x)^2 dx$$

$$= \frac{1}{16} \int \left(\frac{1 - \cos 6x}{2} + \frac{1 - \cos 2x}{2} - \cos 4x + \cos 2x \right) dx$$

$$= \frac{1}{16} \int \left(1 + \frac{1}{2} \cos 2x - \cos 4x - \frac{1}{2} \cos 6x \right) dx = \frac{1}{16} x + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x + C$$

$$24. \int \sin^4 x \cos^2 x dx = \frac{1}{4} \int \sin^2 2x \sin^2 x dx = \frac{1}{16} \int (\cos 3x - \cos x)^2 dx$$

$$= \frac{1}{16} \int \left(\frac{1 + \cos 6x}{2} + \frac{1 + \cos 2x}{2} - \cos 2x - \cos 4x \right) dx$$

$$= \frac{1}{16} \int \left(1 - \frac{1}{2} \cos 2x - \cos 4x + \frac{1}{2} \cos 6x \right) dx = \frac{1}{16} x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x + C$$

$$25. x = a \sin t, \quad dx = a \cos t dt$$

$$\int \sqrt{a^2 - x^2} dx = a^2 \int \cos^2 t dt = \frac{a^2}{2} \int (1 + \cos 2t) dt = \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) + C = \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{1}{2} x \sqrt{a^2 - x^2} + C$$

$$\int \sqrt{1-4x^2} dx = \frac{1}{4} \sin^{-1}(2x) + \frac{1}{2} x \sqrt{1-4x^2} + C$$

26. $x = 2 \tan t, dx = 2 \sec^2 t dt$

$$\int \frac{1}{x^2 \sqrt{x^2+4}} dx = \int \frac{\sec t}{4 \tan^2 t} dx = \frac{1}{4} \int \csc t \cot t dt = -\frac{1}{4} \csc t + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

27. $x+1 = \tan t, dx = \sec^2 t dt$

$$\begin{aligned} \int \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx &= \int \frac{\sec^2 t}{\tan t \sec t} dt = \int \csc t dt = \ln |\csc t - \cot t| + C \\ &= \ln \left| \frac{\sqrt{x^2+2x+2}-1}{x+1} \right| + C \end{aligned}$$

$$\begin{aligned} 28. \int \frac{x^2+2x-2}{x^3-4x} dx &= \int \left(\frac{1}{2x} + \frac{3}{4(x-2)} - \frac{1}{4(x+2)} \right) dx \\ &= \frac{1}{2} \ln |x| + \frac{3}{4} \ln |x-2| - \frac{1}{4} \ln |x+2| + C \end{aligned}$$

$$29. \int \frac{1}{x^4-1} dx = \frac{1}{2} \int \left(\frac{1}{x^2-1} - \frac{1}{x^2+1} \right) dx = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C$$

$$30. \int (x^2+2x+2)^5 (x+1) dx = \frac{1}{12} (x^2+2x+2)^6 + C$$

$$31. \int (x^2-1)\sqrt{x^3-3x} dx = \frac{2}{9} (x^3-3x)^{\frac{3}{2}} + C$$

$$32. \int (1+\sin x)^2 \cos x dx = \frac{1}{3} (1+\sin x)^3 + C$$

$$33. \int \sin x \cos 2x dx = \frac{1}{2} \int (\sin 3x - \sin x) dx = \frac{1}{2} \cos x - \frac{1}{6} \cos 3x + C$$

$$34. \int \frac{e^x}{\sqrt{e^x+1}} dx = 2\sqrt{e^x+1} + C$$

$$35. \int \frac{2x+1}{x^2+x+1} dx = \ln(x^2+x+1) + C$$

$$36. \int \frac{x+1}{x^3-1} dx = \int \left(\frac{2}{3(x-1)} - \frac{2x+1}{3(x^2+x+1)} \right) dx = \frac{2}{3} \ln|x-1| - \frac{1}{3} \ln(x^2+x+1) + C$$

$$37. \int \tan 2x dx = \frac{1}{2} \ln|\sec 2x| + C$$

$$38. \int \frac{1}{x \ln x} dx = \ln|\ln x| + C$$

$$39. \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln(e^x + e^{-x}) + C$$

$$40. t = \sqrt{1+2x}, \quad t^2 = 1+2x, \quad t dt = dx$$

$$\int (2+3x)\sqrt{1+2x} dx = \int t \left(2t + \frac{3}{2}t(t^2-1) \right) dt = \int \left(\frac{1}{2}t^2 + \frac{3}{2}t^4 \right) dt = \frac{3}{10}t^5 + \frac{1}{6}t^3 + C$$

$$= \frac{3}{10}(1+2x)^{\frac{5}{2}} + \frac{1}{6}(1+2x)^{\frac{3}{2}} + C$$

$$41. \int \frac{x-1}{\sqrt{x+1}} dx = \int \left(\sqrt{x+1} - \frac{2}{\sqrt{x+1}} \right) dx = -4(x+1)^{\frac{1}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$$42. t = 1-x^2, \quad dt = -2x dx$$

$$\int \frac{x^3}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{1-t}{\sqrt{t}} dt = -\frac{1}{2} \int \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}} \right) dt = -t^{\frac{1}{2}} + \frac{1}{3}t^{\frac{3}{2}} + C = -(1-x^2)^{\frac{1}{2}} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

$$43. \int \frac{1}{1+\cos^2 x} dx = \int \sec^2 x \cdot \frac{1}{\tan^2 x + 2} dx = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) + C$$

$$44. \int \frac{x}{(2x+1)^3} dx = \frac{1}{2} \int \left(\frac{1}{(2x+1)^2} - \frac{1}{(2x+1)^3} \right) dx = -\frac{1}{4(2x+1)} + \frac{1}{8(2x+1)^2} + C$$

$$45. \int \frac{1}{x^3(x+1)} dx = \int \left(\frac{1}{x^3} - \frac{1}{x^2(x+1)} \right) dx = \int \left(\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \ln \left| \frac{x}{x+1} \right| + \frac{1}{x} - \frac{1}{2x^2} + C$$

$$46. \text{ sol 1) } \int \frac{1}{1 + \cos x} dx = \int \frac{1 - \cos^2 x}{\sin^2 x} dx = \int (\csc^2 x - \csc x \cot x) dx = \csc x - \cot x + C$$

$$\text{sol 2) } \int \frac{1}{1 + \cos x} dx = \int \frac{1}{2 \cos^2 \frac{x}{2}} dx = \frac{1}{2} \int \sec^2 \frac{x}{2} dx = \tan \frac{x}{2} + C$$

$$47. \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$$

$$48. \int \frac{e^x - 1}{e^x + 1} dx = \int \left(\frac{2e^x}{e^x + 1} - 1 \right) dx = 2 \ln(e^x + 1) - x + C$$

$$49. \int \frac{1}{e^{2x} + 1} dx = \int \left(1 - \frac{e^{2x}}{e^{2x} + 1} \right) dx = x - \frac{1}{2} \ln(e^{2x} + 1) + C$$

$$50. \int \frac{1}{e^{2x} + e^x} dx = \int \left(\frac{1}{e^x} - \frac{1}{e^x + 1} \right) dx = \int \left(e^{-x} - 1 + \frac{e^x}{e^x + 1} \right) dx = \ln(e^x + 1) - x - e^{-x} + C$$

$$51. t = e^x + 1, dt = e^x dx$$

$$\int \frac{e^x \ln(e^x + 1)}{e^x + 1} dx = \int \frac{\ln t}{t} dt = \frac{1}{2} \{\ln t\}^2 + C = \frac{1}{2} \{\ln(e^x + 1)\}^2 + C$$

$$52. t = e^x + 3, dt = e^x dx$$

$$\int \frac{e^x (e^x + 1)}{(e^x + 3)^2} dx = \int \frac{t-2}{t^2} dt = \ln|t| + \frac{2}{t} + C = \ln|e^x + 3| + \frac{2}{e^x + 3} + C$$

$$53. t = \ln x + 1, dt = \frac{1}{x} dx$$

$$\int \frac{\ln x}{x(\ln x + 1)^2} dx = \int \frac{t-1}{t^2} dt = \ln|t| + \frac{1}{t} + C = \ln|\ln x + 1| + \frac{1}{\ln x + 1} + C$$

$$54. \int \frac{x^4}{(x-1)^3} dx = \int \left(x + 3 + \frac{6}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3} \right) dx$$

$$= \frac{1}{2} x^2 + 3x + 6 \ln|x-1| - \frac{4}{x-1} - \frac{1}{2(x-1)^2} + C$$

$$55. \int \frac{x^2}{(x^2 - a^2)(x^2 - b^2)} dx = \frac{1}{a^2 - b^2} \int \left(\frac{a^2}{x^2 - a^2} - \frac{b^2}{x^2 - b^2} \right) dx$$

$$= \frac{1}{2(a^2 - b^2)} \left\{ a \ln \left| \frac{x - a}{x + a} \right| - b \ln \left| \frac{x - b}{x + b} \right| \right\} + C$$

$$56. t = x^2 + a, \quad dt = 2x dx$$

$$\int x^3 \sqrt{x^2 + a} dx = \frac{1}{2} \int (t - a) \sqrt{t} dt = \frac{1}{2} \int \left(t^{\frac{3}{2}} - at^{\frac{1}{2}} \right) dt = \frac{1}{5} t^{\frac{5}{2}} - \frac{1}{3} at^{\frac{3}{2}} + C$$

$$= \frac{1}{5} (x^2 + a)^{\frac{5}{2}} - \frac{1}{3} a (x^2 + a)^{\frac{3}{2}} + C$$

$$57. t = \sqrt[4]{x}, \quad dx = 4t^3 dt$$

$$\int \frac{\sqrt{x}}{\sqrt[4]{x^3 + 1}} dx = \int \frac{4t^5}{t^3 + 1} dt = \int \left(4t^2 - \frac{4t^2}{t^3 + 1} \right) dt = \frac{4}{3} t^3 - \frac{4}{3} \ln |t^3 + 1| + C = \frac{4}{3} \sqrt[4]{x^3} - \frac{4}{3} \ln |\sqrt[4]{x^3} + 1| + C$$

$$58. \int \frac{1}{1 + \tan x} dx = \int \frac{\cos x}{\cos x + \sin x} dx = \int \left(\frac{1}{2} + \frac{\cos x - \sin x}{2(\cos x + \sin x)} \right) dx$$

$$= \frac{1}{2} x + \frac{1}{2} \ln |\cos x + \sin x| + C$$

$$59. \int \frac{1}{\sin 2x - \sin x} dx = \int \frac{1}{\sin x (2 \cos x - 1)} dx = \int \frac{-\sin x}{(\cos^2 x - 1)(2 \cos x - 1)} dx$$

$$= \int (-\sin x) \left\{ \frac{1}{2(\cos x - 1)} + \frac{1}{6(\cos x + 1)} - \frac{4}{3(2 \cos x - 1)} \right\} dx$$

$$= \frac{1}{2} \ln |\cos x - 1| + \frac{1}{6} \ln |\cos x + 1| - \frac{2}{3} \ln |2 \cos x - 1| + C$$

$$60. \int \frac{1}{\sin x} dx = \int \csc x dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx = \ln |\csc x - \cot x| + C$$

$$61. \int \frac{1}{1 - 3 \cos^2 x} dx = \int \sec^2 x \cdot \frac{1}{\tan^2 x - 2} dx = \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x - \sqrt{2}}{\tan x + \sqrt{2}} \right| + C$$

$$62. \int \frac{\sin x}{1 + \sin x} dx = \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx = \int (\tan x \sec x - \tan^2 x) dx = \sec x - \tan x + x + C$$

$$63. \quad t = 1 + \sqrt{x}, \quad x = (t-1)^2, \quad dx = 2(t-1)dt$$

$$\begin{aligned} \int \sqrt{1+\sqrt{x}} \, dx &= 2 \int (t-1) \sqrt{t} \, dt = 2 \int (t^{\frac{3}{2}} - t^{\frac{1}{2}}) \, dt = \frac{4}{5} t^{\frac{5}{2}} - \frac{4}{3} t^{\frac{3}{2}} + C \\ &= \frac{4}{5} (1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3} (1+\sqrt{x})^{\frac{3}{2}} + C \end{aligned}$$

$$64. \quad t = x + \sqrt{x^2+3}, \quad x = \frac{t^2-3}{2t}, \quad dx = \frac{t^2+3}{2t^2} dt$$

$$\begin{aligned} \int \sqrt{x + \sqrt{x^2+3}} \, dx &= \frac{1}{2} \int (t^{\frac{1}{2}} - 3t^{-\frac{3}{2}}) \, dt = \frac{1}{3} t^{\frac{3}{2}} - 3t^{-\frac{1}{2}} + C \\ &= \frac{1}{3} (x + \sqrt{x^2+3})^{\frac{3}{2}} - 3(x + \sqrt{x^2+3})^{-\frac{1}{2}} + C \end{aligned}$$

$$\begin{aligned} 65. \quad \int \frac{x}{\sqrt{5-4x-x^2}} \, dx &= \int \frac{x+2}{\sqrt{5-4x-x^2}} \, dx - \int \frac{2}{\sqrt{3^2-(x+2)^2}} \, dx \\ &= -\sqrt{5-4x-x^2} - 2 \sin^{-1} \left(\frac{x+2}{3} \right) + C \end{aligned}$$

$$66. \quad \int \frac{x^3-x-2}{x^3-x^2+x-1} \, dx = \int \left(1 - \frac{1}{x-1} + \frac{2x}{x^2+1} \right) \, dx = x - \ln|x-1| + \ln(x^2+1) + C$$

$$67. \quad \int \frac{1}{x^2+1} \, dx = \tan^{-1} x + C$$

$$68. \quad \int \frac{1}{ax^2+bx+c} \, dx = \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} \, dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right) + C$$

$$69. \quad \int \frac{1}{x^2+x+1} \, dx = \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \, dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$$

$$70. \quad t = \tan \frac{x}{2}, \quad dt = \frac{1}{2} \sec^2 \frac{x}{2} \, dx, \quad dx = 2 \cos^2 \frac{x}{2} \, dt = \frac{2}{t^2+1} \, dt$$

$$\int \frac{1}{3+5\cos x} dx = \int \frac{1}{4-t^2} dt = -\frac{1}{4} \ln \left| \frac{t-2}{t+2} \right| + C = -\frac{1}{4} \ln \left| \frac{\tan \frac{x}{2} - 2}{\tan \frac{x}{2} + 2} \right| + C$$

71. $t = \tan \frac{x}{2}$, $dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$, $dx = 2 \cos^2 \frac{x}{2} dt = \frac{2}{t^2+1} dt$

$$\int \frac{1}{4\sin x + 3\cos x} dx = \int \frac{2}{-3t^2 + 8t + 3} dt = -\frac{1}{5} \ln \left| \frac{t-3}{t+\frac{1}{3}} \right| + C = -\frac{1}{5} \ln \left| \frac{\tan \frac{x}{2} - 3}{\tan \frac{x}{2} + \frac{1}{3}} \right| + C$$

72. $x = a \sin t$, $dx = a \cos t$

$$\int \frac{1}{\sqrt{(a^2 - x^2)^3}} dx = \frac{1}{a^2} \int \sec^2 t dt = \frac{1}{a^2} \tan t + C = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C$$

73. $x = a \tan t$, $dx = a \sec^2 t dt$

$$\int \frac{1}{\sqrt{(a^2 + x^2)^3}} dx = \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C$$

74. $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + C$

75. sol 1) $\int \sin^3 x dx = \int \frac{3\sin x - \sin 3x}{4} dx = -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x + C$

sol 2) $\int \sin^3 x dx = \int (1 - \cos^2 x) \sin x dx = \int \sin x dx - \int \cos^2 x \sin x dx$

$$= -\cos x + \frac{1}{3} \cos^3 x + C$$

76. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

77. $\int \tan^3 x dx = \int \tan x (\sec^2 x - 1) dx = \int \tan x \sec^2 x dx - \int \tan x dx$

$$= \frac{1}{2} \tan^2 x - \ln |\sec x| + C$$

$$78. \int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

$$79. \int \frac{1}{4x^2 + 4x + 2} dx = \int \frac{1}{(2x+1)^2 + 1} dx = \frac{1}{2} \tan^{-1}(2x+1) + C$$

$$80. \int \frac{1}{\sqrt{x^2+1}} dx = \ln |x + \sqrt{x^2+1}| + C$$

$$81. \int \frac{1}{\sqrt{x^2-4}} dx = \ln |x + \sqrt{x^2-4}| + C$$

$$82. \int \frac{1}{\sqrt{x^2-2x+2}} dx = \int \frac{1}{\sqrt{(x-1)^2+1}} dx = \ln |x-1 + \sqrt{x^2-2x+2}| + C$$

$$83. x = 3\sin t, \quad dx = 3\cos t dt$$

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \cot^2 t dt = -\cot t - t + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$84. \int \frac{4x}{(x-1)^2(x+1)} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right) dx = \ln \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1} + C$$

$$85. \int \frac{2x^2-x+4}{x(x^2+4)} dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx = \ln |x| + \frac{1}{2} \ln(x^2+4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$86. \int \frac{1}{16x^4-1} dx = \frac{1}{2} \int \left(\frac{1}{4x^2-1} - \frac{1}{4x^2+1} \right) dx = \frac{1}{8} \ln \left| \frac{2x-1}{2x+1} \right| - \frac{1}{4} \tan^{-1}(2x) + C$$

$$87. \int \frac{x^2+1}{(x-1)(x-2)(x-3)} dx = \int \left(\frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3} \right) dx$$

$$= \ln |x-1| - 5 \ln |x-2| + 5 \ln |x-3| + C$$

$$88. \int \frac{x+4}{x^3+3x^2-10x} dx = \int \frac{x+4}{x(x+5)(x-2)} dx = \int \left(-\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)} \right) dx$$

$$= -\frac{2}{5} \ln |x| - \frac{1}{35} \ln |x+5| + \frac{3}{7} \ln |x-2| + C$$

89. $x = \tan t, dx = \sec^2 t dt$

$$\int \sqrt{x^2 + 1} dx = \int \sec^3 t dt = \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln |\sec t + \tan t| + C \quad (\because 78)$$

$$= \frac{1}{2} x \sqrt{x^2 + 1} + \frac{1}{2} \ln |x + \sqrt{x^2 + 1}| + C$$

90. $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx = \sin^{-1}(e^x) + C$

91. $t = x^2, dt = 2x dx$

$$\int \frac{1}{x(1+x^2)} dx = \int \frac{1}{2t(t+1)} dt = \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt = \frac{1}{2} \ln \left| \frac{t}{t+1} \right| + C = \frac{1}{2} \ln \left| \frac{x^2}{x^2+1} \right| + C$$

92. $x = \tan t, dx = \sec^2 t dt$

$$\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx = \int \frac{1}{\tan^2 t \sec t} \cdot \sec^2 t dt = \int \cot t \operatorname{csc} t dt = -\operatorname{csc} t + C = -\frac{\sqrt{x^2 + 1}}{x} + C$$

93. $\int \frac{1}{x \sqrt{1-x^2}} dx = \int \frac{x}{x^2 \sqrt{1-x^2}} dx = \int \frac{-1}{1-t^2} dt = \frac{1}{2} \ln \left| \frac{1 - \sqrt{1-x^2}}{1 + \sqrt{1-x^2}} \right| + C$

94. $\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2}} dx$

$$= \ln \left| x - \frac{\alpha+\beta}{2} + \sqrt{(x-\alpha)(x-\beta)} \right| + C$$

95. $\int \frac{1}{3\cos^2 x - \sin^2 x} dx = \int \sec^2 x \cdot \frac{1}{3 - \tan^2 x} dx = \frac{1}{2\sqrt{3}} \ln \left| \frac{\tan x + \sqrt{3}}{\tan x - \sqrt{3}} \right| + C$

96. $\int \frac{7x^3 - 13x^2 - 24x + 24}{x^4 - 3x^3 - 10x^2 + 24x} dx = \int \frac{7x^3 - 13x^2 - 24x + 24}{x(x-2)(x+3)(x-4)} dx$

$$= \int \left(\frac{1}{x-2} + \frac{1}{x} + \frac{2}{x+3} + \frac{3}{x-4} \right) dx = \ln |x-2| + \ln |x| + 2 \ln |x+3| + 3 \ln |x-4| + C$$

$$97. t = \sqrt[4]{x}, dt = \frac{1}{4\sqrt[4]{x^3}} dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x} + \sqrt[4]{x^3}} dx &= \int \frac{1}{t^2 + t^3} \cdot 4t^3 dt = 4 \int \frac{t}{t+1} dt = 4 \int \left(1 - \frac{1}{t+1}\right) dt = 4t - 4\ln|t+1| + C \\ &= 4\sqrt[4]{x} - 4\ln|\sqrt[4]{x} + 1| + C \end{aligned}$$

$$98. \text{ sol 1) } t = \frac{1}{x}, dt = -\frac{1}{x^2} dx$$

$$\begin{aligned} I &= \int \frac{1}{x^2 \sqrt{2x-x^2}} dx = \int \frac{1}{x^3 \sqrt{\frac{2}{x}-1}} dx = - \int \frac{t}{\sqrt{2t-1}} dt = -\frac{1}{2} \int \frac{2t-1+1}{\sqrt{2t-1}} dt \\ &= -\frac{1}{2} \int \left(\sqrt{2t-1} + \frac{1}{\sqrt{2t-1}} \right) dt = -\frac{1}{6} (2t-1)^{\frac{3}{2}} - \frac{1}{2} \sqrt{2t-1} + C = -\frac{\sqrt{2t-1}(t+1)}{3} + C \\ &= -\frac{\sqrt{2x-x^2}(x+1)}{3x^2} + C \end{aligned}$$

$$\text{sol 2) } \sin t = x-1, \cos t dt = dx, u = \tan \frac{t}{2}, du = \frac{1}{2} \sec^2 \frac{t}{2} dt$$

$$\begin{aligned} I &= \int \frac{1}{x^2 \sqrt{2x-x^2}} dx = \int \frac{1}{x^2 \sqrt{1-(x-1)^2}} dx = \int \frac{1}{(\sin t + 1)^2 \cos t} \cos t dt \\ &= \int \frac{1}{(\sin t + 1)^2} dt = \int \frac{(u^2+1)^2}{(u+1)^4} \cdot \frac{2}{u^2+1} du = \int \frac{2(u^2+1)}{(u+1)^4} du \\ &= 2 \int \left(\frac{1}{(u+1)^2} - \frac{2}{(u+1)^3} + \frac{2}{(u+1)^4} \right) du = -\frac{2}{u+1} + \frac{2}{(u+1)^2} - \frac{4}{3(u+1)^3} + C \\ &= -\frac{2(3u^2+3u+2)}{3(u+1)^3} + C = -\frac{\sqrt{2x-x^2}(x+1)}{3x^2} + C \end{aligned}$$

$$\ast u = \sqrt{\frac{1 - \sqrt{2x-x^2}}{1 + \sqrt{2x-x^2}}}$$

$$99. x = \sec t, dx = \tan t \sec t dt$$

$$\begin{aligned}
\int \frac{1}{x + \sqrt{x^2 - 1}} dx &= \int (x - \sqrt{x^2 - 1}) dx = \frac{1}{2}x^2 - \int \sqrt{x^2 - 1} dx = \frac{1}{2}x^2 - \int \tan^2 t \operatorname{sect} dt \\
&= \frac{1}{2}x^2 - \int \operatorname{sect}(\sec^2 t - 1) dt = \frac{1}{2}x^2 - \int \sec^3 t dt + \int \operatorname{sect} dt \\
&= \frac{1}{2}x^2 - \frac{1}{2} \operatorname{sect} \tan t + \frac{1}{2} \ln |\operatorname{sect} + \tan t| + C \quad (\because [1]) \\
&= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C
\end{aligned}$$

$$\begin{aligned}
[1] : \int \sec^3 t dt &= \int \operatorname{sect} \cdot \sec^2 t dt = \operatorname{sect} \tan t - \int \operatorname{sect} \tan t \cdot \tan t dt \\
&= \operatorname{sect} \tan t - \int \operatorname{sect}(\sec^2 t - 1) dt = \operatorname{sect} \tan t + \ln |\operatorname{sect} + \tan t| - \int \sec^3 t dt \\
&= \frac{1}{2} \operatorname{sect} \tan t + \frac{1}{2} \ln |\operatorname{sect} + \tan t| + C
\end{aligned}$$

$$100. \quad 2\sin t = x - 1, \quad 2\cos t dt = dx$$

$$\begin{aligned}
\int \frac{1}{(x-1)^2 \sqrt{3+2x-x^2}} dx &= \int \frac{1}{(x-1)^2 \sqrt{4-(x-1)^2}} dx = \int \frac{1}{4\sin^2 t \cdot 2\cos t} \cdot 2\cos t dt \\
&= \frac{1}{4} \int \csc^2 t dt = -\frac{1}{4} \cot t + C = \frac{\sqrt{3+2x-x^2}}{4(1-x)} + C
\end{aligned}$$

$$101. \quad t = x^2 + \frac{1}{x^2} - 1, \quad dt = 2\left(x - \frac{1}{x^3}\right) dx$$

$$\begin{aligned}
\int \frac{x^4 - 1}{x^2 \sqrt{x^4 - x^2 + 1}} dx &= \int \frac{x^4 - 1}{x^3 \sqrt{x^2 + \frac{1}{x^2} - 1}} dx = \int \frac{x - 1/x^3}{\sqrt{x^2 + \frac{1}{x^2} - 1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{t}} dt = \sqrt{t} + C \\
&= \sqrt{x^2 + \frac{1}{x^2} - 1} + C = \frac{\sqrt{x^4 - x^2 + 1}}{x} + C
\end{aligned}$$

$$102. \quad t = x^2, \quad dt = 2x dx$$

$$\int \frac{x^4 + 81}{x(x^2 + 9)^2} dx = \frac{1}{2} \int \frac{t^2 + 81}{t(t+9)^2} dt = \frac{1}{2} \int \left(\frac{1}{t} - \frac{18}{(t+9)^2} \right) dt = \frac{1}{2} \ln |t| + \frac{9}{t+9} + C = \ln |x| + \frac{9}{x^2 + 9} + C$$

$$\begin{aligned}
103. \quad \int \frac{x^2 - x + 2}{x^3 - 1} dx &= \int \left(\frac{x-4}{3(x^2+x+1)} + \frac{2}{3(x-1)} \right) dx = \int \left(\frac{2x+1-9}{6(x^2+x+1)} + \frac{2}{3(x-1)} \right) dx \\
&= \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx - \frac{3}{2} \int \frac{1}{x^2+x+1} dx + \frac{2}{3} \int \frac{1}{x-1} dx \\
&= \frac{1}{6} \ln|x^2+x+1| - \sqrt{3} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + \frac{2}{3} \ln|x-1| + C
\end{aligned}$$

$$\begin{aligned}
104. \quad t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx, \quad t = 2\sin\theta, \quad dt = 2\cos\theta d\theta, \quad \sin\theta = \frac{\sqrt{x}}{2} \\
\int \sqrt{\frac{4-x}{x}} dx = \int \frac{\sqrt{4-t^2}}{t} \cdot 2t dt = 2 \int \sqrt{4-t^2} dt = 2 \int 4\cos^2\theta d\theta = 4 \int (1 + \cos 2\theta) d\theta = 4\theta + 4\sin\theta\cos\theta + C \\
= 4 \left(\sin^{-1} \left(\frac{\sqrt{x}}{2} \right) + \frac{\sqrt{x}}{2} \cdot \frac{\sqrt{4-x}}{2} \right) + C = 4\sin^{-1} \left(\frac{\sqrt{x}}{2} \right) + \sqrt{x(4-x)} + C
\end{aligned}$$

$$\begin{aligned}
105. \quad t = x^{\frac{3}{2}}, \quad dt = \frac{3}{2} \sqrt{x} dx \\
\int \sqrt{\frac{x}{1-x^3}} dx = \frac{2}{3} \int \frac{1}{\sqrt{1-t^2}} dt = \frac{2}{3} \sin^{-1} t + C = \frac{2}{3} \sin^{-1} x^{\frac{3}{2}} + C
\end{aligned}$$

$$\begin{aligned}
106. \quad t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx, \quad t = \sin u, \quad dt = \cos u du \\
\int \sqrt{x} \sqrt{1-x} dx = \int t \sqrt{1-t^2} \cdot 2t dt = 2 \int t^2 \sqrt{1-t^2} dt = 2 \int \sin^2 u \cos u \cdot \cos u du = \frac{1}{2} \int \sin^2 2u du \\
= \frac{1}{4} \int (1 - \cos 4u) du = \frac{1}{4} u - \frac{1}{16} \sin 4u + C = \frac{1}{4} (t \sqrt{1-t^2} (2t^2 - 1) + \sin^{-1} t) + C \\
= \frac{1}{4} (\sqrt{x} \sqrt{1-x} (2x - 1) + \sin^{-1} \sqrt{x}) + C
\end{aligned}$$

$$\begin{aligned}
107. \quad t = \frac{x-2}{x-1}, \quad dt = \frac{1}{(x-1)^2} dx, \quad x = \frac{t-2}{t-1}, \quad t = u^2, \quad dt = 2u du \\
\int \sqrt{\frac{x-2}{x-1}} dx = \int \frac{\sqrt{t}}{(t-1)^2} dt = \int \frac{u}{(u^2-1)^2} \cdot 2u du = 2 \int \frac{u^2}{(u^2-1)^2} du
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \left(-\frac{1}{u+1} + \frac{1}{(u+1)^2} + \frac{1}{u-1} + \frac{1}{(u-1)^2} \right) du = \frac{1}{2} \left(-\frac{2u}{u^2-1} + \ln \left| \frac{1-u}{1+u} \right| \right) + C \\
&= \frac{1}{2} \left(-\frac{2\sqrt{t}}{t-1} + \ln \left| \frac{1-\sqrt{t}}{1+\sqrt{t}} \right| \right) + C = \sqrt{x-1} \sqrt{x-2} + \frac{1}{2} \ln \left| \frac{\sqrt{x-1} - \sqrt{x-2}}{\sqrt{x-1} + \sqrt{x-2}} \right| + C
\end{aligned}$$

108. $x = 5 \sec t$, $dx = 5 \tan t \sec^2 t dt$

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x^3} dx &= \int \frac{5 \tan t}{125 \sec^3 t} \cdot 5 \tan t \sec^2 t dt = \frac{1}{5} \int \sin^2 t dt = \frac{1}{10} \int (1 - \cos 2t) dt \\
&= \frac{1}{10} t - \frac{1}{20} \sin 2t + C = \frac{1}{10} \sec^{-1} \left(\frac{x}{5} \right) - \frac{\sqrt{x^2-25}}{2x^2} + C
\end{aligned}$$

109. $t = \tan x$, $dt = \sec^2 x dx$

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx = \int \frac{t}{1+t^4} dt = \frac{1}{2} \tan^{-1}(t^2) + C = \frac{1}{2} \tan^{-1}(\tan^2 x) + C$$

110. $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \csc^2 x \sec^2 x dx = 4 \int \csc^2 2x dx = -2 \cot 2x + C$

111. $x = \tan t$, $dx = \sec^2 t dt$

$$\begin{aligned}
\int \frac{1}{(1+x^2)^2} dx &= \int \frac{1}{\sec^4 t} \cdot \sec^2 t dt = \int \cos^2 t dt = \frac{1}{2} \int (1 + \cos 2t) dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C \\
&= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{x^2+1} \right) + C
\end{aligned}$$

112. $\int \tan^6 x \sec^4 x dx = \int \tan^6 x (\tan^2 x + 1) \sec^2 x dx = \int \tan^8 x \sec^2 x dx + \int \tan^6 x \sec^2 x dx$

$$= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C$$

113. $\int \tan^4 x \sec^4 x dx = \int \tan^4 x (\tan^2 x + 1) \sec^2 x dx = \int \tan^6 x \sec^2 x dx + \int \tan^4 x \sec^2 x dx$

$$= \frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C$$

$$114. \int \tan x \sec^5 x dx = \int \tan x \sec x \sec^4 x dx = \frac{1}{5} \sec^5 x + C$$

$$115. t = \sec x, dt = \tan x \sec x dx$$

$$\begin{aligned} \int \tan^5 x \sec^7 x dx &= \int \tan x \sec^7 x (\sec^2 x - 1)^2 dx = \int t^6 (t^2 - 1)^2 dt = \int (t^{10} - 2t^8 + t^6) dt \\ &= \frac{t^{11}}{11} - \frac{2t^9}{9} + \frac{t^7}{7} + C = \frac{1}{11} \sec^{11} x - \frac{2}{9} \sec^9 x + \frac{1}{7} \sec^7 x + C \end{aligned}$$

$$116. \int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x + \ln |\sec x| + C$$

$$117. \int x \sin(x+2) dx = -x \cos(x+2) + \int \cos(x+2) dx = -x \cos(x+2) + \sin(x+2) + C$$

$$\begin{aligned} 118. \int \ln(x^2 - x) dx &= \int \{\ln x + \ln(x-1)\} dx = x \ln x - x + (x-1) \ln(x-1) - x + C \\ &= x \ln x + (x-1) \ln(x-1) - 2x + C \end{aligned}$$

$$119. \int e^x \sin^2 \frac{x}{2} dx = \int e^x \left(\frac{1 - \cos x}{2} \right) dx = \frac{1}{2} e^x - \frac{1}{2} \int e^x \cos x dx = \frac{e^x}{4} (2 - \sin x - \cos x) + C$$

$$\begin{aligned} 120. \int \frac{2x+7}{\sqrt{x^2+6x-7}} dx &= \int \left(\frac{1}{\sqrt{(x+3)^2-4^2}} + \frac{2x+6}{\sqrt{x^2+6x+7}} \right) dx \\ &= \ln |x+3 + \sqrt{x^2+6x-7}| + 2\sqrt{x^2+6x-7} + C \end{aligned}$$

$$121. \int \frac{2x-1}{x^2-2x-3} dx = \int \left(\frac{5}{4(x-3)} + \frac{3}{4(x+1)} \right) dx = \frac{5}{4} \ln |x-3| + \frac{3}{4} \ln |x+1| + C$$

$$122. \int 4 \sin 2x \cos 2x \cos 4x dx = \int 2 \sin 4x \cos 4x dx = \int \sin 8x dx = -\frac{1}{8} \cos 8x + C$$

$$123. \int (2x+3)e^x dx = (2x+3)e^x - 2 \int e^x dx = (2x+1)e^x + C$$

$$124. \int x \ln x dx = x(x \ln x - x) - \int (x \ln x - x) dx \quad (\because \int \ln x dx = x \ln x - x + C)$$

$$= x^2 \ln x - x^2 + \frac{1}{2} x^2 - \int x \ln x dx = \frac{1}{4} x^2 (2 \ln x - 1) + C$$

$$125. \int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2x e^x + 2 \int e^x dx = e^x (x^2 - 2x + 2) + C$$

$$126. \int x^2 \ln x dx = \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx = \frac{1}{9} x^3 (3 \ln x - 1) + C$$

$$127. \int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

$$128. \int (\ln x)^2 dx = x (\ln x)^2 - \int \frac{2 \ln x}{x} \cdot x dx = x (\ln x)^2 - 2 \int \ln x dx = x \{ (\ln x)^2 - 2 \ln x + 2 \} + C$$

$$129. \int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x (\sin x - \cos x) - \int e^x \sin x dx$$

$$= \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$130. \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx = e^x (\sin x + \cos x) - \int e^x \cos x dx$$

$$= \frac{1}{2} e^x (\sin x + \cos x) + C$$

$$131. \int e^x \sin^2 x dx = \frac{1}{2} \int e^x (1 - \cos 2x) dx = \frac{1}{2} e^x - \frac{1}{5} e^x (\sin 2x + \frac{1}{2} \cos 2x) + C$$

$$= \frac{1}{10} e^x (5 - 2 \sin 2x - \cos 2x) + C$$

$$132. \textcircled{1} n \neq -1, \int x^n \ln x dx = \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n dx$$

$$= \frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1} + C = \frac{x^{n+1}}{(n+1)^2} \{ (n+1) \ln x - 1 \} + C$$

$$\textcircled{2} n = -1, \int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + C$$

$$133. \int \ln x dx = x \ln x - x + C$$

$$134. \int \ln(x+1) dx = (x+1) \ln(x+1) - x + C$$

$$135. \int \frac{1}{1+e^x} dx = \int \left(1 - \frac{e^x}{1+e^x}\right) dx = x - \ln(1+e^x) + C$$

$$136. \int \frac{x^3+x^2+1}{x^2(x^2+1)} dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2}\right) dx = \frac{1}{2} \ln|x^2+1| - \frac{1}{x} + C$$

$$137. \int (x^2+1)e^x dx = e^x + x^2e^x - \int 2xe^x dx = (x^2+1)e^x - 2xe^x + 2 \int e^x dx$$

$$= e^x(x^2 - 2x + 3) + C$$

$$138. t = \sqrt{x+1}, \quad dt = \frac{1}{2\sqrt{x+1}} dx$$

$$\int \frac{1}{x\sqrt{x+1}} dx = \int \frac{2}{t^2-1} dt = - \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt = \ln \left| \frac{t-1}{t+1} \right| + C = \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$$

$$139. t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx$$

$$\int \frac{x\sqrt{x}-1}{x-\sqrt{x}} dx = \int \frac{t^3-1}{t^2-t} \cdot 2t dt = 2 \int (t^2+t+1) dt = \frac{2}{3}t^3 + t^2 + 2t + C = \frac{2}{3}x\sqrt{x} + x + 2\sqrt{x} + C$$

$$140. \int \ln(x + \sqrt{x^2+1}) dx = x \ln(x + \sqrt{x^2+1}) - \int x \cdot \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} dx$$

$$= x \ln(x + \sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}} dx = x \ln(x + \sqrt{x^2+1}) - \sqrt{x^2+1} + C$$

$$141. \int \left(x^2 - \frac{1}{x^2}\right) \left(x - \frac{1}{x}\right) dx = \int \left(x^3 - x - \frac{1}{x} + \frac{1}{x^3}\right) dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 - \ln|x| - \frac{1}{2x^2} + C$$

$$142. t = \sqrt{x}, \quad dt = \frac{1}{2\sqrt{x}} dx$$

$$\int \sin \sqrt{x} dx = 2 \int t \sin t dt = -2t \cos t + 2 \int \cos t dt = 2 \sin t - 2t \cos t + C = 2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C$$

$$143. \int \ln(x^2+1) dx = x \ln(x^2+1) - \int \frac{2x^2}{x^2+1} dx = x \ln(x^2+1) - \int \left(2 - \frac{2}{x^2+1}\right) dx$$

$$= x\{\ln(x^2 + 1) - 2\} + 2\tan^{-1}x + C$$

144. $t = x^2$, $dt = 2x dx$

$$\int \frac{-2x}{\sqrt{1-x^4}} dx = - \int \frac{1}{\sqrt{1-t^2}} dt = -\sin^{-1}t + C = -\sin^{-1}(x^2) + C$$

145. $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}} dx$

$$\int e^{\sqrt{x}} dx = 2 \int te^t dt = 2te^t - 2 \int e^t dt = 2e^t(t-1) + C = 2e^{\sqrt{x}}(\sqrt{x}-1) + C$$

146. $x = a \sin t$, $dx = a \cos t dt$

$$\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \int \frac{a^2 \sin^2 t}{a \cos t} \cdot a \cos t dt = a^2 \int \sin^2 t dt = \frac{a^2}{2} \int (1 - \cos 2t) dt = \frac{a^2}{4} (2t - \sin 2t) + C$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{x\sqrt{a^2-x^2}}{2} + C$$

147. $x = a \sin t$, $dx = a \cos t dt$

$$\int \frac{1}{x\sqrt{a^2-x^2}} dx = \int \frac{a \cos t}{a \sin t \cdot a \cos t} dt = \frac{1}{a} \int \operatorname{csc} t dt = \frac{1}{a} \ln |\operatorname{csc} t - \cot t| + C$$

$$= \frac{1}{a} \ln \left| \frac{a - \sqrt{a^2-x^2}}{x} \right| + C$$

148. $t = e^x$, $dt = e^x dx$, $u = \sqrt{t+1}$, $du = \frac{1}{2\sqrt{t+1}} dt$

$$\int \frac{1}{\sqrt{1+e^x}} dx = \int \frac{1}{t\sqrt{t+1}} dt = \int \frac{2}{(u^2-1)} du = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{t+1}-1}{\sqrt{t+1}+1} \right| + C$$

$$= \ln \left| \frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1} \right| + C$$

149. $t = x^2$, $dt = 2x dx$, $t = \tan u$, $dt = \sec^2 u du = (1+t^2) du$

$$\int \frac{x}{\sqrt{1+x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{t^2+1}} dt = \frac{1}{2} \int \sec u du = \frac{1}{2} \ln |\tan u + \sec u| + C = \frac{1}{2} \ln |t + \sqrt{t^2+1}| + C$$

$$= \frac{1}{2} \ln |x^2 + \sqrt{x^4+1}| + C$$

150. $t = \ln x$, $dt = \frac{1}{x} dx$, $t = \sin u$, $dt = \cos u du$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx = \int \frac{\sqrt{1-t^2}}{t} dt = \int \frac{\cos^2 u}{\sin u} du = \int \frac{1-\sin^2 u}{\sin u} du = \int (\csc u - \sin u) du$$

$$= \ln |\csc u - \cot u| + \cos u + C = \ln \left| \frac{1-\sqrt{1-t^2}}{t} \right| + \sqrt{1-t^2} + C = \ln \left| \frac{1-\sqrt{1-(\ln x)^2}}{\ln x} \right| + \sqrt{1-(\ln x)^2} + C$$

151. $\int \frac{x+2}{(x^2+1)(x-1)^3} dx = \int \left(\frac{3-x}{4(x^2+1)} + \frac{1}{4(x-1)} - \frac{1}{(x-1)^2} + \frac{3}{2(x-1)^3} \right) dx$

$$= -\frac{1}{8} \int \frac{2x}{x^2+1} dx + \frac{3}{4} \int \frac{1}{x^2+1} dx + \frac{1}{4} \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx + \frac{3}{2} \int \frac{1}{(x-1)^3} dx$$

$$= -\frac{1}{8} \ln |x^2+1| + \frac{3}{4} \tan^{-1} x + \frac{1}{4} \ln |x-1| + \frac{1}{x-1} - \frac{3}{4(x-1)^2} + C$$

152. $x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan t$, $dx = \frac{\sqrt{3}}{2} \sec^2 t dt$

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}} dx = 2 \int \frac{\sec t}{\sqrt{3} \tan t + 1} dt = 2 \int \frac{1}{\sqrt{3} \sin t + \cos t} dt$$

$$= \int \sec \left(t - \frac{\pi}{3} \right) dt = \ln \left| \sec \left(t - \frac{\pi}{3} \right) + \tan \left(t - \frac{\pi}{3} \right) \right| + C$$

$$= \ln \left| \frac{\sin t - \sqrt{3} \cos t + 2}{\sqrt{3} \sin t + \cos t} \right| + C = \ln \left| \frac{x-1+2\sqrt{x^2+x+1}}{\sqrt{3}(x+1)} \right| + C$$

153. $x = \tan t$, $dx = \sec^2 t dt = (1+x^2) dt$

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \int t dt = \frac{1}{2} t^2 + C = \frac{1}{2} (\tan^{-1} x)^2 + C$$

154. $x = 2 \sin t$, $dx = 2 \cos t dt$

$$\int \frac{4}{x^2 \sqrt{4-x^2}} dx = 4 \int \frac{1}{4 \sin^2 t} dt = \int \csc^2 t dt = -\cot t + C = -\frac{\sqrt{4-x^2}}{x} + C$$

155. $t = 3x, dt = 3dx$

$$\int 3 \sec^4 3x dx = \int \sec^4 t dt = \int \sec^2 t (1 + \tan^2 t) dt = \int \sec^2 t dt + \int \sec^2 t \tan^2 t dt$$

$$= \tan t + \frac{1}{3} \tan^3 t + C = \tan 3x + \frac{1}{3} \tan^3 3x + C$$

156. $\int \cot^3 x dx = \int \cot x (\csc^2 x - 1) dx = -\frac{1}{2} \cot^2 x - \ln |\sin x| + C$

157. $\int \csc^4 x dx = \int \csc^2 x (1 + \cot^2 x) dx = \int \csc^2 x dx - \int \cot^2 x \cdot (-\csc^2 x) dx$

$$= -\cot x - \frac{1}{3} \cot^3 x + C$$

158. $t = 3x, dt = 3dx$

$$\int \tan^2 3x dx = \frac{1}{3} \int \tan^2 t dt = \frac{1}{3} \int (\sec^2 t - 1) dt = \frac{1}{3} \tan t - \frac{1}{3} t + C = \frac{1}{3} \tan 3x - x + C$$

159. $\int \frac{3x^4 - 3x^3 - x^2 - 17x - 2}{(x-3)(x^2+1)^2} dx = \int \left(\frac{2x+3}{x^2+1} + \frac{4x-2}{(x^2+1)^2} + \frac{1}{x-3} \right) dx$

$$= \int \frac{2x}{x^2+1} dx + 3 \int \frac{1}{x^2+1} dx + 2 \int \frac{2x-1}{(x^2+1)^2} dx + \int \frac{1}{x-3} dx$$

$$= \ln |x^2+1| + 3 \tan^{-1} x + 2 \left(-\frac{1}{2} \tan^{-1} x + \frac{x^2-x-1}{2(x^2+1)} \right) + \ln |x-3| + C$$

$$= \ln |(x^2+1)(x-3)| + 2 \tan^{-1} x + \frac{x^2-x-1}{x^2+1} + C$$

$$= \ln |(x^2+1)(x-3)| + 2 \tan^{-1} x - \frac{x+2}{x^2+1} + C$$

$$\ast x = \tan t, dx = \sec^2 t dt, \int \frac{2x-1}{(x^2+1)^2} dx = \int \frac{2 \tan t - 1}{\sec^4 t} \cdot \sec^2 t dt = \int \frac{2 \tan t - 1}{\sec^2 t} dt$$

$$= \int (2\sin t \cos t - \cos^2 t) dt = \int \left(\sin 2t - \frac{1 + \cos 2t}{2} \right) dt$$

$$= -\frac{1}{2} \cos 2t - \frac{1}{2} t - \frac{1}{4} \sin 2t + C = -\frac{1}{2} \tan^{-1} x + \frac{x^2 - x - 1}{2(x^2 + 1)} + C$$

$$160. \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx = \int \left(-\frac{1}{10(x+2)} + \frac{1}{5(2x-1)} + \frac{1}{2x} \right) dx$$

$$= \frac{1}{5} \int \frac{1}{2x-1} dx - \frac{1}{10} \int \frac{1}{x+2} dx + \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{10} \ln |2x-1| - \frac{1}{10} \ln |x+2| + \frac{1}{2} \ln |x| + C$$

$$161. x = \sin t, dx = \cos t dt$$

$$\int \frac{1}{x - \sqrt{1-x^2}} dx = \int \frac{\cos t}{\sin t - \cos t} dt = \frac{1}{2} \int \left(\frac{\cos t + \sin t}{\sin t - \cos t} - 1 \right) dt = \frac{1}{2} \ln |\cos t - \sin t| - \frac{1}{2} t + C$$

$$= \frac{1}{2} \ln |\sqrt{1-x^2} - x| - \frac{1}{2} \sin^{-1} x + C$$

$$162. \int \frac{x^2 + 3}{x^2 - 1} dx = \int \left(1 + \frac{2}{x-1} - \frac{2}{x+1} \right) dx = x + 2 \ln \left| \frac{x-1}{x+1} \right| + C$$

$$163. I = \int 13e^{2x} \cos 3x dx = \frac{13}{2} e^{2x} \cos 3x + \frac{3}{2} \int 13e^{2x} \sin 3x dx$$

$$= \frac{13}{2} e^{2x} \cos 3x + \frac{39}{2} \left\{ \frac{1}{2} e^{2x} \sin 3x - \frac{1}{2} \int 3e^{2x} \cos 3x dx \right\} = \frac{13}{4} e^{2x} (2 \cos 3x + 3 \sin 3x) - \frac{9}{4} \int 13e^{2x} \cos 3x dx$$

$$= \frac{13}{4} e^{2x} (2 \cos 3x + 3 \sin 3x) - \frac{9}{4} I$$

$$\therefore I = e^{2x} (2 \cos 3x + 3 \sin 3x) + C$$

$$164. \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 + 4x} dx = \int \left(\frac{16 - 25x}{4(x^2 + 4)} + x + \frac{1}{4x} \right) dx$$

$$= -\frac{25}{8} \int \frac{2x}{x^2 + 4} dx + 4 \int \frac{1}{x^2 + 4} dx + \int x dx + \frac{1}{4} \int \frac{1}{x} dx$$

$$= -\frac{25}{8} \ln|x^2+4| + 2 \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x^2 + \frac{1}{4} \ln|x| + C$$

$$165. \int \frac{2x^2-x+4}{x^3+4x} dx = \int \left(\frac{x-1}{x^2+4} + \frac{1}{x} \right) dx = \frac{1}{2} \int \frac{2x}{x^2+4} dx - \int \frac{1}{x^2+4} dx + \int \frac{1}{x} dx$$

$$= \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \ln|x| + C$$

$$166. \int \frac{1}{(x+1)(x^2+1)} dx = \int \left(\frac{1-x}{2(x^2+1)} + \frac{1}{2(x+1)} \right) dx$$

$$= \frac{1}{2} \int \frac{1}{x^2+1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x+1} dx = \frac{1}{2} \tan^{-1}x - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \ln|x+1| + C$$

$$167. \int \frac{x^2-2x-2}{x^3-1} dx = \int \left(\frac{2x+1}{x^2+x+1} - \frac{1}{x-1} \right) dx = \ln|x^2+x+1| - \ln|x-1| + C$$

$$168. x = a \sin t, dx = a \cos t dt$$

$$\int \sqrt{a^2-x^2} dx = a^2 \int \cos^2 t dt = \frac{a^2}{2} \int (1 + \cos 2t) dt = \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) + C = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} x \sqrt{a^2-x^2} + C$$

$$169. t = \tan x, \sec^2 x dx = dt$$

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx = \int \frac{1}{a^2 t^2 + b^2} dt = \frac{1}{a^2} \int \frac{1}{t^2 + \frac{b^2}{a^2}} dt = \frac{1}{ab} \tan^{-1}\left(\frac{a}{b}t\right) + C$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan x\right) + C$$

$$170. \int \frac{1}{\sin(x-a)\sin(x-b)} dx = \frac{1}{\sin(b-a)} \int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \left(\frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} \right) dx = \frac{1}{\sin(b-a)} \int (\cot(x-b) - \cot(x-a)) dx$$

$$= \frac{\ln|\sin(x-b)| - \ln|\sin(x-a)|}{\sin(b-a)} + C$$

$$171. \int x \sec^2 x \tan x dx = \frac{1}{2} x \sec^2 x - \frac{1}{2} \int \sec^2 x dx = \frac{1}{2} x \sec^2 x - \frac{1}{2} \tan x + C$$

$$\begin{aligned}
172. \int e^{-x} \sin^2 2x dx &= \frac{1}{2} \int e^{-x} (1 - \cos 4x) dx = -\frac{1}{2} e^{-x} - \frac{1}{2} \int e^{-x} \cos 4x dx \quad \cdots [1] \\
&= -\frac{1}{2} e^{-x} - \frac{1}{2} \left\{ -e^{-x} \cos 4x - 4 \int e^{-x} \sin 4x dx \right\} = -\frac{1}{2} e^{-x} (1 - \cos 4x) + 2 \int e^{-x} \sin 4x dx \\
&= -\frac{1}{2} e^{-x} (1 - \cos 4x) + 2 \left\{ -e^{-x} \sin 4x + 4 \int e^{-x} \cos 4x dx \right\} \\
&= -\frac{1}{2} e^{-x} (1 - \cos 4x + 4 \sin 4x) + 8 \int e^{-x} \cos 4x dx \quad \cdots [2]
\end{aligned}$$

[1], [2]에서

$$\int e^{-x} \cos 4x dx = -\frac{1}{17} e^{-x} (\cos 4x - 4 \sin 4x) + C$$

$$\therefore \int e^{-x} \sin^2 2x dx = -\frac{1}{2} e^{-x} - \frac{1}{2} \int e^{-x} \cos 4x dx = \frac{1}{34} e^{-x} (\cos 4x - 4 \sin 4x - 17) + C$$

$$173. \text{ sol 1) } t^2 = \tan x, \quad 2t dt = \sec^2 x dx, \quad dx = \frac{2t}{t^4 + 1} dt$$

$$\int \sqrt{\tan x} dx = \int \frac{2t^2}{t^4 + 1} dt = \int \frac{2}{t^2 + \frac{1}{t^2}} dt = \int \frac{\left(1 + \frac{1}{t^2}\right) + \left(1 - \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} dt = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt + \int \frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 2} dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2} dt + \int \frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - (\sqrt{2})^2} dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t - 1/t}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{t + 1/t - \sqrt{2}}{t + 1/t + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{\tan x} + \sqrt{\cot x} - \sqrt{2}}{\sqrt{\tan x} + \sqrt{\cot x} + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sin x - \cos x}{\sqrt{\sin 2x}} \right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\sin x + \cos x - \sqrt{\sin 2x}}{\sin x + \cos x + \sqrt{\sin 2x}} \right| + C$$

$$\text{sol 2) } I = \int \sqrt{\tan x} dx, \quad J = \int \sqrt{\cot x} dx$$

$$I + J = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \sqrt{2} \int \frac{(\sin x - \cos x)'}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

$$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C_1 \quad \dots \quad [1]$$

$$I - J = \int (\sqrt{\tan x} - \sqrt{\cot x}) dx = \sqrt{2} \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = -\sqrt{2} \int \frac{(\sin x + \cos x)'}{\sqrt{(\sin x + \cos x)^2 - 1}} dx$$

$$= -\sqrt{2} \ln |(\sin x + \cos x) + \sqrt{(\sin x + \cos x)^2 - 1}| + C_2 \quad \dots \quad [2]$$

$$[1] + [2] : I = \int \sqrt{\tan x} dx = \frac{1}{\sqrt{2}} \sin^{-1}(\sin x - \cos x) - \frac{1}{\sqrt{2}} \ln |\sin x + \cos x + \sqrt{\sin 2x}| + C$$

$$174. \int \sqrt[3]{\tan x} dx$$

$$[1] \quad t^3 = \tan x, \quad x = \tan^{-1}(t^3), \quad dx = \frac{3t^2}{1+t^6} dt$$

$$[2] \quad a = t^2, \quad da = 2t dt, \quad a = \tan^{2/3} x$$

$$\int \sqrt[3]{\tan x} dx = \int \frac{3t^3}{1+t^6} dt \quad \dots \quad [1]$$

$$= \frac{3}{2} \int \frac{a}{1+a^3} da \quad \dots \quad [2]$$

$$= \frac{3}{2} \int \left(-\frac{1}{3(a+1)} + \frac{a+1}{3(a^2-a+1)} \right) da = \frac{1}{2} \int \left(\frac{2a-1}{2(a^2-a+1)} + \frac{3}{2(a^2-a+1)} - \frac{1}{a+1} \right) da$$

$$= \frac{1}{4} \ln \frac{a^2-a+1}{(a+1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(a - \frac{1}{2} \right) \right) + C$$

$$= \frac{1}{4} \ln \frac{\tan^{4/3} x - \tan^{2/3} x + 1}{(\tan^{2/3} x + 1)^2} + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2}{\sqrt{3}} \left(\tan^{2/3} x - \frac{1}{2} \right) \right) + C$$

$$175. \int \left(\frac{x+3}{\sqrt{4-x^2}} + \cot x [\ln(\sin x)] \right) dx = \int \left(\frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}} + \frac{\cos x}{\sin x} [\ln(\sin x)] \right) dx$$

$$= -\sqrt{4-x^2} + 3\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}[\ln(\sin x)]^2 + C$$

$$176. \int \left(\frac{4x^2}{x^2+9} + \tan^2 2x \right) dx = \int \left(3 - \frac{36}{x^2+9} + \sec^2 2x \right) dx = 3x - 12 \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2} \tan 2x + C$$

$$177. \int \frac{1}{(1+x)(2+x)^2(3+x)^3} dx$$

$$= \int \left(\frac{2}{x+2} - \frac{17}{8(x+3)} - \frac{1}{(x+2)^2} - \frac{5}{4(x+3)^2} - \frac{1}{2(x+3)^3} + \frac{1}{8(x+1)} \right) dx$$

$$= 2\ln|x+2| - \frac{17}{8}\ln|x+3| + \frac{1}{x+2} + \frac{5}{4(x+3)} + \frac{1}{4(x+3)^2} + \frac{1}{8}\ln|x+1| + C$$

$$178. \cot x = t^2, \quad -\csc^2 x dx = 2t dt, \quad dx = -\frac{2t}{1+t^4} dt$$

$$\int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int \frac{1}{1 + \sqrt{\cot x}} dx = \int \frac{1}{1+t} \cdot \frac{-2t}{1+t^4} dt = -\int \frac{2t}{(1+t)(1+t^4)} dt$$

$$= -\int \left(\frac{t}{1+t^4} + \frac{t^3}{1+t^4} + \frac{1-t^2}{1+t^4} - \frac{1}{1+t} \right) dt$$

$$= -\int \left(\frac{t}{1+t^4} + \frac{t^3}{1+t^4} - \frac{1-t^{-2}}{(t+t^{-1})^2-2} - \frac{1}{1+t} \right) dt$$

$$= -\int \left(\frac{t}{1+t^4} + \frac{t^3}{1+t^4} - \frac{(t+t^{-1})'}{(t+t^{-1})^2-2} - \frac{1}{1+t} \right) dt$$

$$= -\frac{1}{2} \tan^{-1}(t^2) - \frac{1}{4} \ln|1+t^4| + \frac{1}{2\sqrt{2}} \ln \left| \frac{t+t^{-1}-\sqrt{2}}{t+t^{-1}+\sqrt{2}} \right| + \ln|1+t| + C$$

$$= -\frac{1}{2} \tan^{-1}(\cot x) - \frac{1}{4} \ln|1+\cot^2 x| + \frac{1}{2\sqrt{2}} \ln \left| \frac{\cot x - \sqrt{2\cot x + 1}}{\cot x + \sqrt{2\cot x + 1}} \right| + \ln|1+\sqrt{\cot x}| + C$$

$$179. \tan x = t, \quad \sec^2 x dx = dt, \quad dt = \frac{1}{1+t^2} dt$$

$$\int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx = \int \frac{t+t^3}{1+t^3} \cdot \frac{1}{1+t^2} dt = \int \frac{t}{1+t^3} dt = \frac{1}{2} \int \frac{t+1+t-1}{t^3+1} dt$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{t+1}{(t+1)(t^2-t+1)} dt - \frac{1}{2} \int \frac{t^2-t+1-t^2}{t^3+1} dt = \frac{1}{2} \int \frac{1}{t^2-t+1} dt - \frac{1}{2} \int \frac{1}{t+1} dt + \frac{1}{6} \int \frac{3t^2}{t^3+1} dt \\
&= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2t-1}{\sqrt{3}} \right) - \frac{1}{2} \ln|t+1| + \frac{1}{6} \ln|t^3+1| + C \\
&= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2\tan x-1}{\sqrt{3}} \right) - \frac{1}{2} \ln|\tan x+1| + \frac{1}{6} \ln|\tan^3 x+1| + C
\end{aligned}$$

180. $t = \tan x, dt = \sec^2 x dx$

$$\begin{aligned}
\int \frac{1}{\cos 2x+3} dx &= \int \frac{1}{\frac{1-\tan^2 x}{1+\tan^2 x}+3} dx = \int \frac{1+\tan^2 x}{2\tan^2 x+4} dx = \frac{1}{2} \int \frac{\sec^2 x}{\tan^2 x+2} dx = \frac{1}{2} \int \frac{1}{t^2+2} dt \\
&= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + C = \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) + C
\end{aligned}$$

$$\begin{aligned}
181. \int e^x \tan x (1-2\sec^2 x) dx &= \int e^x \tan x dx - \int e^x \cdot 2\tan x \sec^2 x dx \\
&= \int e^x \tan x dx - \int e^x \cdot (\sec^2 x)' dx = \left\{ e^x \tan x - \int e^x \sec^2 x dx \right\} - \left\{ e^x \sec^2 x - \int e^x \sec^2 x dx \right\} \\
&= e^x (\tan x - \sec^2 x) + C
\end{aligned}$$

182. $t = \sqrt{\tan x+1}, dt = \frac{\sec^2 x}{2\sqrt{\tan x+1}} dx = \frac{(t^2-1)^2+1}{2t} dx$

$$\begin{aligned}
\int \sqrt{\tan x+1} dx &= \int \frac{2t^2}{(t^2-1)^2+1} dt = 2 \int \frac{t^2}{t^4-2t^2+2} dt = 2 \int \frac{1}{t^2+\frac{2}{t^2}-2} dt \\
&= \int \frac{1+\sqrt{2}/t}{t^2+\frac{2}{t^2}-2} dt + \int \frac{1-\sqrt{2}/t}{t^2+\frac{2}{t^2}-2} dt \\
&= \int \frac{d\left(t-\frac{\sqrt{2}}{t}\right)}{(t-\sqrt{2}/t)^2+2(\sqrt{2}-1)} + \int \frac{d\left(t+\frac{\sqrt{2}}{t}\right)}{(t+\sqrt{2}/t)^2-2(\sqrt{2}+1)} \\
&= \frac{1}{\sqrt{2}(\sqrt{2}-1)} \tan^{-1} \left(\frac{t-\sqrt{2}/t}{\sqrt{2}(\sqrt{2}-1)} \right) + \frac{1}{2\sqrt{2}(\sqrt{2}+1)} \ln \left| \frac{t+\sqrt{2}/t-\sqrt{2}(\sqrt{2}+1)}{t+\sqrt{2}/t+\sqrt{2}(\sqrt{2}+1)} \right| + C
\end{aligned}$$

$$= \frac{1}{\sqrt{2(\sqrt{2}-1)}} \tan^{-1}\left(\frac{\tan x + 1 - \sqrt{2}}{\sqrt{2(\sqrt{2}-1)}(\tan x + 1)}\right) + \frac{1}{2\sqrt{2(\sqrt{2}+1)}} \ln \left| \frac{\tan x + 1 - \sqrt{2(\sqrt{2}+1)}(\tan x + 1) + \sqrt{2}}{\tan x + 1 + \sqrt{2(\sqrt{2}+1)}(\tan x + 1) + \sqrt{2}} \right| + C$$

$$183. \quad t = \sqrt{\tan x + 2}, \quad dt = \frac{\sec^2 x}{2\sqrt{\tan x + 2}} dx = \frac{(t^2 - 2)^2 + 1}{2t} dx$$

$$I = \int \sqrt{\tan x + 2} dx = \int \frac{2t^2}{(t^2 - 2)^2 + 1} dt = 2 \int \frac{t^2}{t^4 - 4t^2 + 5} dt$$

$$= 2 \int \frac{t^2}{(t^2 + \sqrt{4 + \sqrt{20}}t + \sqrt{5})(t^2 - \sqrt{4 + \sqrt{20}}t + \sqrt{5})} dt$$

$$= 2 \int \left(\frac{At}{t^2 + \sqrt{4 + \sqrt{20}}t + \sqrt{5}} - \frac{Bt}{t^2 - \sqrt{4 + \sqrt{20}}t + \sqrt{5}} \right) dt.$$

$$A = -B = -\frac{1}{4} \sqrt{4 + \sqrt{20}} (-2 + \sqrt{5}).$$

$$\text{한편 } \int \frac{t}{t^2 + bt + c} dt = \frac{1}{2} \ln |t^2 + bt + c| - \frac{b}{\sqrt{4c - b^2}} \tan^{-1} \left(\frac{2t + b}{\sqrt{4c - b^2}} \right) + C, \quad 4c - b^2 > 0 \text{ 이고}$$

이때 $4c - b^2 = 4\sqrt{5} - (4 + \sqrt{20}) = 2\sqrt{5} - 4 > 0$ 이므로

$$\int \frac{t}{t^2 + \sqrt{4 + \sqrt{20}}t + \sqrt{5}} dt = \frac{1}{2} \ln |t^2 + \sqrt{4 + \sqrt{20}}t + \sqrt{5}| - \frac{\sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \tan^{-1} \left(\frac{2t + \sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \right) + C$$

$$\int \frac{t}{t^2 - \sqrt{4 + \sqrt{20}}t + \sqrt{5}} dt = \frac{1}{2} \ln |t^2 - \sqrt{4 + \sqrt{20}}t + \sqrt{5}| + \frac{\sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \tan^{-1} \left(\frac{2t - \sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \right) + C$$

이다. 따라서

$$I = 2A \left\{ \frac{1}{2} \ln |t^2 + \sqrt{4 + \sqrt{20}}t + \sqrt{5}| - \frac{\sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \tan^{-1} \left(\frac{2t + \sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \right) \right\} \\ - 2A \left\{ \frac{1}{2} \ln |t^2 - \sqrt{4 + \sqrt{20}}t + \sqrt{5}| + \frac{\sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \tan^{-1} \left(\frac{2t - \sqrt{4 + \sqrt{20}}}{\sqrt{2\sqrt{5} - 4}} \right) \right\} + C.$$

$t = \sqrt{\tan x + 2}$ 를 다시 대입하여 정리하면

$$\begin{aligned} \therefore I &= \frac{(2 - \sqrt{5})\sqrt{4 + \sqrt{20}}}{4} \ln \left| \frac{\tan x + 2 + \sqrt{4 + \sqrt{20}} \sqrt{\tan x + 2} + \sqrt{5}}{\tan x + 2 - \sqrt{4 + \sqrt{20}} \sqrt{\tan x + 2} + \sqrt{5}} \right| \\ &+ \frac{\sqrt{4 + \sqrt{20}}}{2} \left\{ \tan^{-1} \left(\frac{2\sqrt{\tan x + 2} - \sqrt{4 + \sqrt{20}}}{\sqrt{\sqrt{20} - 4}} \right) + \tan^{-1} \left(\frac{2\sqrt{\tan x + 2} + \sqrt{4 + \sqrt{20}}}{\sqrt{\sqrt{20} - 4}} \right) \right\} + C \end{aligned}$$

184. $x = \tan t, dx = \sec^2 t dt$

$$\int \frac{\tan^{-1} x}{x\sqrt{x^2+1}} \cdot \exp\left(-\frac{\tan^{-1} x}{x}\right) dx = \int \frac{t}{\tan t \sec t} \cdot \exp\left(-\frac{t}{\tan t}\right) \sec^2 t dt = \int \frac{t}{\sin t} e^{-t \cot t} dt$$

$= e^{-t \cot t} \cdot u(t) + C$ 라 하면

$$t \csc t e^{-t \cot t} = (e^{-t \cot t} \cdot u(t))' = (t \csc^2 t - \cot t) e^{-t \cot t} \cdot u(t) + e^{-t \cot t} \cdot u'(t)$$

$$t \csc t = (t \csc^2 t - \cot t) u(t) + u'(t)$$

$$t \sin t = (t - \cot t \sin t) u(t) + u'(t) \sin^2 t$$

에서 $u(t) = \sin t$ 가 근이 된다.

$$e^{-t \cot t} \cdot u(t) + C = e^{-t \cot t} \cdot \sin t + C = \exp\left(-\frac{\tan^{-1} x}{x}\right) \cdot \frac{x}{\sqrt{1+x^2}} + C$$

185. $x = \cot t, \sin(\cot^{-1} x) = \sin t = \frac{x}{\sqrt{1+x^2}}, x = \tan u, \cos(\tan^{-1} x) = \cos u = \frac{1}{\sqrt{1+x^2}}$

$$\int \cos^2(\tan^{-1}(\sin(\cot^{-1} x))) dx = \int \cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) dx = \int \frac{1+x^2}{2+x^2} dx$$

$$= \int \left(1 - \frac{1}{2+x^2}\right) dx = x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

186. $t = \sin x, dt = \cos x dx, u = \frac{1}{t}, dt = -\frac{1}{u^2} du, w = \sqrt{1-u}, dw = -\frac{1}{2\sqrt{1-u}} du$

$$\int \frac{\sec x - \tan x}{\sqrt{\sin^2 x - \sin x}} dx = \int \frac{1 - \sin x}{\cos x \sqrt{\sin^2 x - \sin x}} dx = \int \frac{(1 - \sin x) \cos x}{(1 - \sin^2 x) \sqrt{\sin^2 x - \sin x}} dx$$

$$= \int \frac{\cos x}{(1 + \sin x) \sqrt{\sin^2 x - \sin x}} dx = \int \frac{1}{(1+t) \sqrt{t^2 - t}} dt = - \int \frac{1}{(u+1) \sqrt{1-u}} du$$

$$= 2 \int \frac{1}{2-w^2} dw = \frac{1}{\sqrt{2}} \ln \left| \frac{w + \sqrt{2}}{w - \sqrt{2}} \right| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{1 - \csc x} + \sqrt{2}}{\sqrt{1 - \csc x} - \sqrt{2}} \right| + C$$

187. $t = \tan x$, $dt = \sec^2 x dx$, $u = t + \sqrt{1+t^2}$, $t = \frac{1}{2} \left(u - \frac{1}{u} \right)$, $dt = \frac{1}{2} \left(1 + \frac{1}{u^2} \right) du$

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{5/2}} dx = \int \frac{1}{(t + \sqrt{1+t^2})^{5/2}} dt = \frac{1}{2} \int (u^{-5/2} + u^{-9/2}) du = -\frac{1}{3} u^{-3/2} - \frac{1}{7} u^{-7/2} + C$$

$$= -\frac{1}{3} (\sec x + \tan x)^{-3/2} - \frac{1}{7} (\sec x + \tan x)^{-7/2} + C$$

188. $t = \sec x + \tan x$, $dt = \sec x (\sec x + \tan x) dx$

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx = \frac{1}{2} \int \frac{\sec x (\sec x + \tan x) + \sec x (\sec x - \tan x)}{(\sec x + \tan x)^{9/2}} dx$$

$$= \frac{1}{2} \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)^{9/2}} dx + \frac{1}{2} \int \frac{\sec x (\sec x - \tan x)}{(\sec x + \tan x)^{9/2}} dx$$

$$= \frac{1}{2} \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)^{9/2}} dx + \frac{1}{2} \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)^{13/2}} dx$$

$$= \frac{1}{2} \int t^{-9/2} dt + \frac{1}{2} \int t^{-13/2} dt$$

$$= -\frac{1}{7} t^{-9/2} - \frac{1}{11} t^{-11/2} + C$$

$$= -\frac{1}{7} (\sec x + \tan x)^{-9/2} - \frac{1}{11} (\sec x + \tan x)^{-11/2} + C$$

189. $\int \sin 2022x \cdot \sin^{2020} x dx = \int \sin(2021x + x) \cdot \sin^{2020} x dx$

$$= \int \sin 2021x \cdot \cos x \cdot \sin^{2020} x dx + \int \cos 2021x \cdot \sin^{2021} x dx$$

$$= \sin 2021x \cdot \frac{1}{2021} \sin^{2021} x - \frac{2021}{2021} \int \cos 2021x \cdot \sin^{2021} x dx + \int \cos 2021x \cdot \sin^{2021} x dx$$

$$= \frac{1}{2021} \sin 2021x \cdot \sin^{2021} x dx + C$$

190. sol 1) $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$, $t^2 = 1 - \tan^2 x$, $2t dt = -2 \tan x \sec^2 x dx$

$$\int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \frac{\sqrt{1 - \tan^2 x}}{\tan x} dx = \int \frac{t}{\tan x} \cdot \frac{-t}{\tan x \sec^2 x} dt = - \int \frac{t^2}{(1 - t^2)(2 - t^2)} dt$$

$$= - \int \left(\frac{1}{1 - t^2} - \frac{2}{2 - t^2} \right) dt = - \frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| + \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}+t}{\sqrt{2}-t} \right| + C$$

$$= - \frac{1}{2} \ln \left| \frac{1 + \sqrt{1 - \tan^2 x}}{1 - \sqrt{1 - \tan^2 x}} \right| + \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + C$$

sol 2) $u = \frac{\cos x}{\sqrt{\cos 2x}}$, $du = \frac{-\sin x \sqrt{\cos 2x} + \cos x \cdot \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos 2x} dx$

$$= \frac{-\sin x \cos 2x + \cos x \sin 2x}{(\sqrt{\cos 2x})^3} dx = \frac{\sin x}{(\sqrt{\cos 2x})^3} dx$$

$$\int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \frac{\cos^2 2x}{\sin^2 x} du = \int \frac{\cos 2x}{\cos^2 x - \cos^2 x + \sin^2 x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos^2 x + \sin^2 x} du$$

$$= \int \frac{\cos 2x}{\cos^2 x - \cos 2x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos 2x} du = \int \frac{1}{(u^2 - 1)(2u^2 - 1)} du$$

$$= \int \left(\frac{1}{u^2 - 1} - \frac{2}{2u^2 - 1} \right) du = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}u-1}{\sqrt{2}u+1} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\cos x - \sqrt{\cos 2x}}{\cos x + \sqrt{\cos 2x}} \right| - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} \cos x - \sqrt{\cos 2x}}{\sqrt{2} \cos x + \sqrt{\cos 2x}} \right| + C$$

sol 3) $t = \tan \frac{x}{2}$, $dt = \frac{1}{2} \sec^2 \frac{x}{2} dx = \frac{t^2 + 1}{2} dx$, $u = t^2$, $du = 2t dt$

$$y - u = \sqrt{u^2 - 6u + 1}, \quad u = \frac{y^2 - 1}{2y - 6} \quad (\text{오일러 치환})$$

$$\begin{aligned}
\int \frac{\sqrt{\cos 2x}}{\sin x} dx &= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x} dx = \int \frac{\sqrt{t^4 - 6t^2 + 1}}{t^3 + t} dt = \frac{1}{2} \int \frac{\sqrt{u^2 - 6u + 1}}{u^2 + u} du \\
&= \frac{1}{2} \int \frac{(y^2 - 6y + 1)^2}{(y-1)(y-3)(y+1)(y^2 + 2y - 7)} dy \\
&= \frac{1}{2} \int \left(\frac{1}{y-1} + \frac{1}{y-3} - \frac{1}{y+1} - \frac{16}{(y+1)^2 - 8} \right) dy \\
&= \frac{1}{2} \ln|y-1| + \frac{1}{2} \ln|y-3| - \frac{1}{2} \ln|y+1| - \sqrt{2} \ln \left| \frac{y-2\sqrt{2}+1}{y+2\sqrt{2}+1} \right| + C \\
&= \frac{1}{2} \ln \left| \frac{(y-1)(y-3)}{y+1} \right| - \sqrt{2} \ln \left| \frac{y-2\sqrt{2}+1}{y+2\sqrt{2}+1} \right| + C
\end{aligned}$$

$$(y = \tan^2 \frac{x}{2} + \sqrt{\tan^4 \frac{x}{2} - 6 \tan^2 \frac{x}{2} + 1})$$

191. 제 1종 오일러 치환 : $\sqrt{x^2 + 4x - 4} = x + t$, $x = \frac{t^2 + 4}{4 - 2t}$, $dx = \frac{-2t^2 + 8t + 8}{(4 - 2t)^2} dt$

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2 + 4x - 4}} dx &= \int \frac{4 - 2t}{t^2 + 4} \cdot \frac{4 - 2t}{-t^2 + 4t + 4} \cdot \frac{-2t^2 + 8t + 8}{(4 - 2t)^2} dt = 2 \int \frac{1}{t^2 + 4} dt \\
&= \tan^{-1} \left(\frac{t}{2} \right) + C = \tan^{-1} \left(\frac{\sqrt{x^2 + 4x - 4} - x}{2} \right) + C
\end{aligned}$$

192. 제 2종 오일러 치환 : $\sqrt{-x^2 + x + 2} = xt + \sqrt{2}$, $x = \frac{1 - 2\sqrt{2}t}{t^2 + 1}$, $dx = \frac{2\sqrt{2}t^2 - 2t - 2\sqrt{2}}{(t^2 + 1)^2} dt$

$$\begin{aligned}
\int \frac{1}{x\sqrt{-x^2 + x + 2}} dx &= \int \frac{t^2 + 1}{1 - 2\sqrt{2}t} \cdot \frac{t^2 + 1}{-\sqrt{2}t^2 + t + \sqrt{2}} \cdot \frac{2\sqrt{2}t^2 - 2t - 2\sqrt{2}}{(t^2 + 1)^2} dt \\
&= \int \frac{-2}{-2\sqrt{2}t + 1} dt = \frac{1}{\sqrt{2}} \int \frac{-2\sqrt{2}}{-2\sqrt{2}t + 1} dt = \frac{1}{\sqrt{2}} \ln|2\sqrt{2}t - 1| + C \\
&= \frac{1}{\sqrt{2}} \ln \left| \frac{2\sqrt{-2x^2 + 2x + 4} - 4}{x} - 1 \right| + C
\end{aligned}$$

193. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)} = (x-2)t$, $x = \frac{-2t^2 - 1}{-t^2 - 1}$, $dx = \frac{2t}{(-t^2 - 1)^2} dt$

$$\int \frac{x^2}{\sqrt{-x^2+3x-2}} dx = \int \left(\frac{-2t^2-1}{-t^2-1} \right)^2 \cdot \frac{-t^2-1}{t} \cdot \frac{2t}{(-t^2-1)^2} dt = -2 \int \frac{(2t^2+1)^2}{(t^2+1)^3} dt$$

$$= -2 \int \left(\frac{4}{t^2+1} - \frac{4}{(t^2+1)^2} + \frac{1}{(t^2+1)^3} \right) dt = -8 \tan^{-1} t + 8I - 2J$$

$$= \frac{19}{8} \tan^{-1} t - \frac{t(13t^2+11)}{8(t^2+1)^2} + C = \frac{19}{8} \tan^{-1} \left(\frac{\sqrt{-(x-2)(x-1)}}{x-2} \right) + \frac{(2x+9)\sqrt{-(x-1)(x-2)}}{8} + C$$

$$\ast \quad x = \tan t, \quad dx = \sec^2 t dt, \quad I = \int \frac{1}{(x^2+1)^2} dx = \int \cos^2 t dt = \frac{1}{2} \int (1 + \cos 2t) dt$$

$$= \frac{1}{2} t + \frac{1}{4} \sin 2t + C = \frac{1}{2} \tan^{-1} x + \frac{x}{2(1+x^2)} + C$$

$$\ast\ast \quad x = \tan t, \quad dx = \sec^2 t dt, \quad J = \int \frac{1}{(x^2+1)^3} dx = \int \cos^4 t dt = \frac{1}{4} \int (1 + \cos 2t)^2 dt$$

$$= \frac{1}{4} \int (1 + 2\cos 2t + \cos^2 2t) dt = \frac{1}{4} \int \left(1 + 2\cos 2t + \frac{1 + \cos 4t}{2} \right) dt$$

$$= \frac{1}{8} \int (3 + 4\cos 2t + \cos 4t) dt = \frac{3}{8} t + \frac{1}{4} \sin 2t + \frac{1}{32} \sin 4t + C = \frac{3}{8} \tan^{-1} x + \frac{x(3x^2+5)}{8(x^2+1)^2} + C$$

$$194. \quad t = \cos \theta, \quad dt = -\sin \theta d\theta, \quad u = \frac{t-1}{t+1}, \quad t = \frac{1+u}{1-u}, \quad dt = \frac{2}{(1-u)^2} du$$

$$\int \frac{\sin^3(\theta/2)}{\cos(\theta/2) \cdot \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta = \frac{1}{2} \int \frac{(1 - \cos \theta) \sin \theta}{(1 + \cos \theta) \sqrt{\cos^3 \theta + \cos^2 \theta + \cos \theta}} d\theta$$

$$= \frac{1}{2} \int \frac{t-1}{(t+1) \sqrt{t^3+t^2+t}} dt$$

$$= \frac{1}{2} \int u \cdot \frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3 + (1+u)^2(1-u) + (1+u)(1-u)^2}} \cdot \frac{2}{(1-u)^2} du$$

$$= \int \frac{u}{\sqrt{3-2u^2-u^4}} du = \frac{1}{2} \int \frac{2u}{\sqrt{4-(u^2+1)^2}} du = \frac{1}{2} \sin^{-1} \left(\frac{u^2+1}{2} \right) + C = \frac{1}{2} \sin^{-1} \left(\frac{t^2+1}{(t+1)^2} \right) + C$$

$$= \frac{1}{2} \sin^{-1} \left(\frac{\cos^2 \theta + 1}{(\cos \theta + 1)^2} \right) + C = \frac{1}{2} \sin^{-1} \left[\frac{1}{4} \sec^4 \left(\frac{\theta}{2} \right) - \tan^2 \left(\frac{\theta}{2} \right) \right] + C$$

$$\begin{aligned} 195. \quad & \int \frac{\tan^4 \theta}{1 - \tan^2 \theta} d\theta = \int \frac{\tan^4 \theta - 1}{1 - \tan^2 \theta} d\theta + \int \frac{1}{1 - \tan^2 \theta} d\theta \\ & = - \int (\tan^2 \theta + 1) d\theta + \int \frac{\cos^2 \theta}{\cos^2 \theta - \sin^2 \theta} d\theta = - \int \sec^2 \theta d\theta + \int \frac{1 + \cos 2\theta}{2 \cos 2\theta} d\theta \\ & = - \tan \theta + \int \left(\frac{1}{2} \sec 2\theta + \frac{1}{2} \right) d\theta = - \tan \theta + \frac{1}{4} \ln |\sec 2\theta + \tan 2\theta| + \frac{1}{2} \theta + C \end{aligned}$$

$$196. \quad t = x - \frac{1}{x}, \quad dt = \left(1 + \frac{1}{x^2} \right) dx, \quad \frac{dx}{x} = \frac{x}{x^2 + 1} dt$$

$$\begin{aligned} I &= \int \frac{x^2 + 1}{x^4 + 3x^3 + 3x^2 - 3x + 1} dx = \int \frac{x + x^{-1}}{(x^2 + x^{-2}) + 3(x - x^{-1}) + 3} \cdot \frac{dx}{x} \\ &= \int \frac{1}{t^2 + 3t + 5} dt = \int \frac{1}{\left(t + \frac{3}{2}\right)^2 + \frac{11}{4}} dt = \sqrt{\frac{4}{11}} \tan^{-1} \left(\sqrt{\frac{4}{11}} \left(t + \frac{3}{2} \right) \right) + C \\ &= \frac{2}{\sqrt{11}} \tan^{-1} \left(\frac{2}{\sqrt{11}} \left(x - \frac{1}{x} + \frac{3}{2} \right) \right) + C. \end{aligned}$$

cf) 위 표현은 $x = 0$ 에서 불연속이다. 따라서 정적분을 할 때에는 $x > 0$ 인 구간과 $x < 0$ 인 구간의 적분상수를 각각 다르게 설정해야 한다. 이들 적분상수를 각각 C_+ , C_- 라 하면, 위 적분은 $x = 0$ 에서 well-behaved하므로 그 부정적분 I 는 $x = 0$ 에서 연속이다. 즉,

$$C_+ - C_- = \frac{2\pi}{\sqrt{11}}$$

이다. 이를 이용하면

$$I = \frac{2}{\sqrt{11}} \left[\tan^{-1} \left(\frac{2}{\sqrt{11}} \left(x - \frac{1}{x} + \frac{3}{2} \right) \right) + \delta(x) \right] + C'$$

$$\left(C' = \frac{C_+ + C_-}{2}, \quad \delta(x) = \begin{cases} +\frac{\pi}{2} & (x > 0) \\ -\frac{\pi}{2} & (x < 0) \end{cases} \right)$$

이다. 이를 간단히하기 위해

$$\delta(x) = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right), \quad x \neq 0$$

이러 하면,

$$\begin{aligned} I &= \frac{2}{\sqrt{11}} \left[\tan^{-1}\left(\frac{2}{\sqrt{11}}\left(x - \frac{1}{x} + \frac{3}{2}\right)\right) + \tan^{-1}\left(\frac{2}{\sqrt{11}x}\right) + \tan^{-1}\left(\frac{\sqrt{11}x}{2}\right) \right] + C' \\ &= \frac{2}{\sqrt{11}} \left[\tan^{-1}\left(\frac{\sqrt{11}x^2(2x+3)}{7x^2-6x+4}\right) + \tan^{-1}\left(\frac{\sqrt{11}x}{2}\right) \right] + C' \end{aligned}$$

이고 이때 I 는 $\forall x \in \mathbb{R}$ 에 대해 연속이다.

$$197. \quad t = \sqrt{x+1+x^{-1}}, \quad dt = \frac{d(x+1+x^{-1})}{2\sqrt{x+1+x^{-1}}}$$

$$\begin{aligned} \int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx &= \int \frac{x-1}{(x+1)\sqrt{x+1+x^{-1}}} \frac{dx}{x} \\ &= \int \frac{x-1}{(x+1)\sqrt{x+1+x^{-1}}} \frac{d(x+1+x^{-1})}{x-x^{-1}} = \int \frac{d(x+1+x^{-1})}{(x+2+x^{-1})\sqrt{x+1+x^{-1}}} = 2 \int \frac{1}{1+t^2} dt \\ &= 2 \tan^{-1}t + C = 2 \tan^{-1}\sqrt{x+1+x^{-1}} + C \end{aligned}$$

$$\begin{aligned} 198. \quad \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}} dx &= \int \left(\frac{e^x}{(1-x)\sqrt{1-x^2}} + e^x \sqrt{\frac{1+x}{1-x}} \right) dx \\ &= \int \left(e^x \left(\sqrt{\frac{1+x}{1-x}} \right)' + (e^x)' \sqrt{\frac{1+x}{1-x}} \right) dx = e^x \sqrt{\frac{1+x}{1-x}} + C \end{aligned}$$

$$199. \quad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \text{ 이므로}$$

$$\int \frac{u'v - v'u}{v^2} dx = \int \left(\frac{u}{v}\right)' dx = \frac{u}{v} + C$$

이다. 이때

$$I = \int \left(\frac{\tan^{-1}x}{1+(x+1/x)\tan^{-1}x} \right)^2 dx = \int \frac{x^2(\tan^{-1}x)^2}{(x+(x^2+1)\tan^{-1}x)^2} dx$$

이고, $v = x + (x^2 + 1)\tan^{-1}x$ 라 하면 $v' = 2 + 2x\tan^{-1}x$ 이다. 이제

$$u'v - v'u = u'[x + (x^2 + 1)\tan^{-1}x] - u[2 + 2x\tan^{-1}x] = x^2(\tan^{-1}x)^2 \cdots [1]$$

인 함수 $u(x)$ 를 찾으려면 된다. 만약

$$u = \frac{-x^2 + (x^2 + 1)(\tan^{-1}x)^2}{2}$$

이면

$$u' = -x + x(\tan^{-1}x)^2 + \tan^{-1}x$$

이고 이 u 는 기적적으로 [1]식을 만족시킨다. 즉,

$$I = \int \left(\frac{-x^2 + (1+x^2)(\tan^{-1}x)^2}{2[x + (x^2 + 1)\tan^{-1}x]} \right)' dx = \frac{-x^2 + (1+x^2)(\tan^{-1}x)^2}{2[x + (x^2 + 1)\tan^{-1}x]} + C.$$

$$200. \int \frac{1}{\prod_{i=0}^m (x+i)} dx = \int \frac{1}{x(x+1)(x+2)\cdots(x+m)} dx \quad (m \in \mathbb{N} \cup \{0\})$$

sol 1) $q_m(x) := \prod_{k=0}^m (x+k)$, $f_m(x) := \frac{1}{q_m(x)}$ 라 하자. 실수 A_i ($0 \leq i \leq m$)에 대하여 f_m 을 부분분수의 합으로 분해하면

$$f_m(x) = \sum_{n=0}^m \frac{A_n}{x+n}$$

이다. 따라서 f_m 의 부정적분은

$$\int f_m(x) dx = \sum_{n=0}^m A_n \ln|x+n| + C$$

이다. 또한

$$1 = f_m(x)q_m(x) = q_m(x) \sum_{n=0}^m \frac{A_n}{x+n} = \sum_{n=0}^m A_n p_n(x)$$

라 하면,

$$p_n(x) = \prod_{k \neq n} (x+k) = (-1)^n \prod_{k=0}^{n-1} (-x-k) \cdot \prod_{k=n+1}^m (k+x)$$

이다. 이때

$$p_n(-n) = (-1)^n \prod_{k=0}^{n-1} (n-k) \cdot \prod_{k=n+1}^m (k-n) = (-1)^n n! (m-n)! = \frac{(-1)^n m!}{\binom{m}{n}}$$

이고 $n \neq k \leq m$ 인 음이 아닌 정수 k 에 대하여

$$p_n(-k) = 0$$

이다. 따라서

$$A_n = \frac{1}{p_n(-n)} = \frac{(-1)^n}{m!} \binom{m}{n}$$

이고

$$\int f_m(x) dx = \sum_{n=0}^m \frac{(-1)^n}{m!} \binom{m}{n} \ln|x+n| + C.$$

sol 2) $\frac{1}{\prod_{i=0}^m (x+i)} = \frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{j=0}^m \frac{a_j}{x+j}$... [1] 이라 하자.

해비사이드법에 의해

$$a_j = \lim_{x \rightarrow (-j)} \frac{x+j}{x(x+1)(x+2)\cdots(x+m)}$$

이다. 이때

$$a_1 = \frac{1}{(-1)(2)(3)\cdots(m-1)} = \frac{(-1)^1}{(m-1)!},$$

$$a_2 = \frac{1}{(-2)(-1)(1)(2)\cdots(m-2)} = \frac{(-1)^2}{2!(m-2)!},$$

$$a_3 = \frac{1}{(-3)(-2)(-1)(1)\cdots(m-3)} = \frac{(-1)^3}{3!(m-3)!}$$

이고 수학적 귀납법을 이용하면

$$a_j = \frac{(-1)^j}{j!(m-j)!} = \frac{(-1)^j}{m!} \binom{m}{j}$$

임이 증명된다. 따라서 [1]의 양변을 적분하면

$$\begin{aligned} \int \frac{1}{x(x+1)(x+2)\cdots(x+m)} dx &= \sum_{j=0}^m \int \frac{a_j}{x+j} dx, \\ \int \frac{1}{x(x+1)(x+2)\cdots(x+m)} dx &= \sum_{j=0}^m a_j \ln|x+j| + C \\ &= \sum_{j=0}^m \frac{(-1)^j}{m!} \binom{m}{j} \ln|x+j| + C. \end{aligned}$$