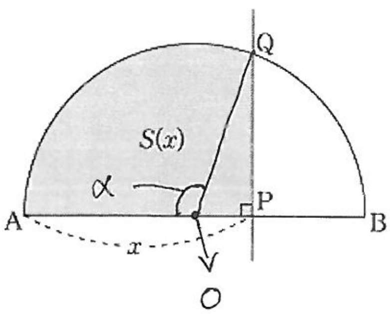
* 2017년 10월 시행 교육청 모의고사 고3수학 가형 30번.



$\overline{AB}=2$. $\overline{AO}=\overline{OB}=r=1$, $\angle AOQ=\alpha$ $(0<\alpha<\pi)$

(문제에서 주어진 θ와는 다름. $0<x<2$ 이므로 $0<\alpha<\pi$)

$x = 1+\cos(\pi-\alpha) = 1-\cos\alpha$.

$\therefore S(x) =$ 부채꼴 + 삼각형 $= \frac{1}{2}\times 1^2 \times \alpha + \frac{1}{2}\times\cos(\pi-\alpha)\times\sin(\pi-\alpha) = \frac{1}{2}\alpha - \frac{1}{2}\cos\alpha\cdot\sin\alpha$

$= \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha$. ($0<\alpha<\frac{\pi}{2}$, $\frac{\pi}{2}\le\alpha<2\pi$ 나눠서도 정리해 볼 것)

$\therefore S(x) = S(1-\cos\alpha) = \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha$.

(i) $S(1+\sin\theta)$는 $\theta=\frac{\pi}{2}$에서 선대칭이므로 $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} S(1+\sin\theta)\,d\theta = 2\cdot\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} S(1+\sin\theta)\,d\theta$

이때 이미 정리된 $S(1-\cos\alpha)$ 형태를 사용하려면 $\alpha = \frac{\pi}{2}+\theta$로 치환한다.

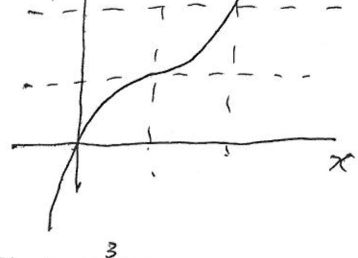
$$\left.\begin{aligned}&\theta=\tfrac{\pi}{4} \to \alpha=\tfrac{3\pi}{4},\quad d\alpha=d\theta\\ &\theta=\tfrac{\pi}{2} \to \alpha=\pi,\end{aligned}\right\}\ \therefore 2\cdot\int_{\frac{3\pi}{4}}^{\pi} S(1-\cos\alpha)\,d\alpha \leftarrow 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} S(1+\sin\theta)\,d\theta$$

$$\therefore 2\int_{\frac{3\pi}{4}}^{\pi} S(1-\cos\alpha)\,d\alpha = 2\cdot\int_{\frac{3}{4}\pi}^{\pi}\left\{\frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha\right\}d\alpha = 2\times\left[\frac{1}{4}\alpha^2 + \frac{1}{8}\cos 2\alpha\right]_{\frac{3}{4}\pi}^{\pi}$$

$$= 2\times\left[\left(\frac{\pi^2}{4} - \frac{9\pi^2}{64}\right) + \frac{1}{8}(1-0)\right] = \frac{7}{32}\pi^2 + \frac{1}{4}.$$

(ii) $S(1+\cos\theta)$는 $\theta=\frac{\pi}{2}$에서 점대칭이므로 $\theta=\frac{\pi}{2}$일 때 피적분함수 $S(1+\cos\theta)$의 값을 적분구간만큼 곱한 값과 같다. $\to \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} S(1+\cos\theta)\,d\theta = S(1+\cos\frac{\pi}{2})\times\frac{\pi}{2}$.

ex)



$y=(x-1)^3+1$

$\int_0^2 y\,dx = y(1)\times(2-0)$

$= 2$.

$\theta=\frac{\pi}{2}$일 때 $S(1+\cos\theta)=S(1)$. $\alpha=\frac{\pi}{2}$일 때이므로

$S(1-\cos\alpha) = S(1) = \frac{1}{2}\times\left(\frac{\pi}{2}\right) - \frac{1}{4}\times\sin\left(2\times\frac{\pi}{2}\right) = \frac{\pi}{4}$.

$$\therefore \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} S(1+\cos\theta)\,d\theta = S(1)\times\frac{\pi}{2} = \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}$$

따라서 $\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\{S(1+\sin\theta)-S(1+\cos\theta)\}d\theta = \frac{7\pi^2}{32}+\frac{1}{4}-\frac{\pi^2}{8} = \frac{3\pi^2}{32}+\frac{1}{4}$. $\therefore p=\frac{1}{4}.\ q=\frac{3}{32}$.

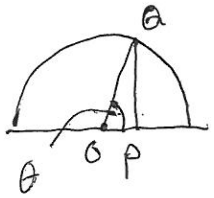
$$\therefore \frac{30p}{q} = \frac{\frac{1}{4}\times 30}{\frac{3}{32}} = \frac{240}{3} = 80$$

→ 치환적분을 할 때 x의 정의역에 맞춰서 적분구간을 조절해야 한다.

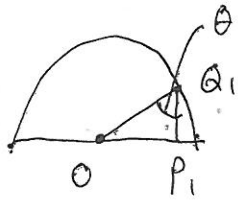
* Version 2.

$S(x)$에서 변수 x는 길이. $S(1+\sin\theta)$와 $S(1+\cos\theta)$를 $x(=\overline{AP})$로 나타내보면.

$\angle QOP=\theta$라 할 때 (단, O는 $\overline{AB}$의 중점)

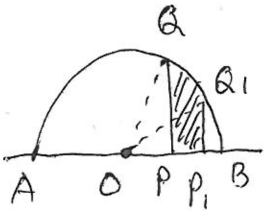


$\overline{OP}=\cos\theta$.

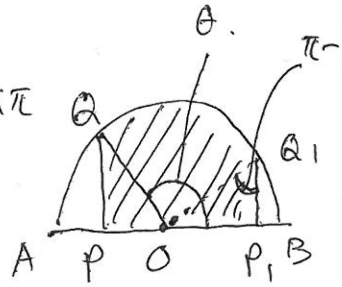


$\overline{OP_1}=\sin\theta$.

$\therefore$ (i) $0<\theta<\frac{\pi}{2}$



(ii) $\frac{\pi}{2}\le\theta<\pi$

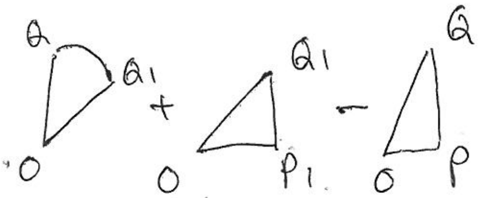


$\therefore \angle Q_1OP_1=\theta-\frac{\pi}{2}$.

$\angle QOQ_1=\frac{\pi}{2}$.

빗금친 부분이 $S(1+\sin\theta)-S(1+\cos\theta)$ 이므로

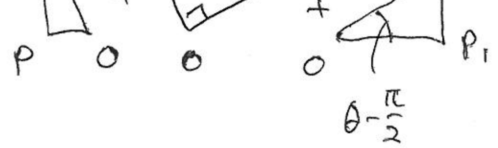
(i)



$$=\frac{1}{2}\times1^2\times\left(\theta-\left(\frac{\pi}{2}-\theta\right)\right)+\frac{1}{2}\times\sin\theta\times1\times\sin\left(\frac{\pi}{2}-\theta\right)-\frac{1}{2}\cos\theta\cdot\sin\theta$$

$$=\theta-\frac{\pi}{4}.$$

(ii)



$$=\frac{1}{2}\times1\times\cos(\pi-\theta)\times\sin(\pi-\theta)+\frac{1}{2}\times1\times1\times\frac{\pi}{2}$$

$$+\frac{1}{2}\times1\times\cos\left(\theta-\frac{\pi}{2}\right)\times\sin\left(\theta-\frac{\pi}{2}\right)=\frac{\pi}{4}-\sin\theta\cos\theta$$

$$=\frac{\pi}{4}-\frac{1}{2}\sin2\theta.$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left\{\theta-\frac{\pi}{4}\right\}d\theta+\int_{\frac{\pi}{2}}^{\frac{3}{4}\pi}\left\{\frac{\pi}{4}-\frac{1}{2}\sin2\theta\right\}d\theta$$

$$=\left[\frac{\theta^2}{2}-\frac{\pi}{4}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}+\left[\frac{\pi}{4}\theta+\frac{\cos2\theta}{4}\right]_{\frac{\pi}{2}}^{\frac{3}{4}\pi}=\left(\frac{\pi^2}{8}-\frac{\pi^2}{8}\right)-\left(\frac{\pi^2}{32}-\frac{\pi^2}{16}\right)+\left(\frac{3}{16}\pi^2+0\right)-\left(\frac{\pi}{8}-\frac{1}{4}\right)=\frac{3\pi^2}{32}+\frac{1}{4}.$$